Algebraic Methods for Computing Picard Groups

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Let R be a commutative ring.

Instead of trying to study R by itself, one might instead study $\mathrm{Mod}(R)$, the category of modules over R.

In $\mathrm{Mod}(R)$, we have an operation called tensor product, denoted $\otimes_R$ or $\otimes$, which satisfies the following properties:

1. It has a unit, given by R: $M \otimes_R R \cong M \cong R \otimes_R M$.
2. It is associative: $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$.
3. It is symmetric: $M \otimes N \cong N \otimes M$.

Given an R-module N, we have a functor

$$- \otimes_R N : \mathrm{Mod}(R) \to \mathrm{Mod}(R)$$

Question: When is $- \otimes N : \mathrm{Mod}(R) \to \mathrm{Mod}(R)$ an equivalence of categories?

Theorem

The following are equivalent:

(i) $- \otimes N : \mathrm{Mod}(R) \to \mathrm{Mod}(R)$ *is an equivalence of categories.*

(ii) *There exists an R-module M such that $M \otimes N \cong R$. We say that N is invertible.*

(iii) *N is finitely generated projective of rank 1.*

In fact, in case (ii) we have that $M \cong \mathrm{Hom}_R(N, R)$.

Observation: The set of isomorphism classes of invertible R-modules has a group structure:

Definition

The Picard group of R, denoted $\mathrm{Pic}(R)$, is the set of isomorphism classes of invertible modules, with

$$[M] \cdot [N] = [M \otimes N]$$

$$[M]^{-1} = [\mathrm{Hom}_R(M, R)]$$

Example

For R a local ring or PID, $\mathrm{Pic}(R)$ is trivial.

Proof.

For local rings/PIDs, a module is projective iff it is free. Hence $M \in \mathrm{Pic}(R)$ iff M is a free rank 1 R-module. □

Chain Complexes of R-modules

Let's see what happens if we work with chain complexes of R-modules, $\mathrm{Ch}(R)$, instead.

Definition

The tensor product of two chain complexes $X_\bullet$ and $Y_\bullet$ is defined at degree n by

$$(X \otimes Y)_n = \oplus_{i+j=n} X_i \otimes Y_j$$

This tensor product is also associative and symmetric, and has unit given by $R[0]$.

Question: When is $Y_\bullet$ invertible?

Theorem

The following are equivalent for a local ring R:

(i) *$Y_\bullet$ is invertible. That is, there exists a chain complex $X_\bullet$ such that $X_\bullet \otimes Y_\bullet \cong R[0]$.*

(ii) *$Y_\bullet$ is the chain complex $R[n]$, that is, the complex R concentrated in a single degree n.*

Example

For R a local ring, $\mathrm{Pic}(\mathrm{Ch}(R))$ is isomorphic to $\mathbb{Z}$.

To define $\mathrm{Pic}(R)$ and $\mathrm{Pic}(\mathrm{Ch}(R))$ we only really needed the associative, symmetric, and unital structure of $\otimes$.

Definition

Suppose we have a category $\mathcal{C}$ that has bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with unit 1 and is associative and symmetric.
Then we say that $(\mathcal{C}, \otimes, 1)$ is a **symmetric monoidal category**.

Example

The following categories are symmetric monoidal:

(a) $(\mathrm{Set}, \times, \{*\})$
(b) $(\mathrm{Group}, \times, \{e\})$
(c) $(\mathrm{Mod}(R), \otimes, R)$

Definition

The Picard group of a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$, denoted $\text{Pic}(\mathcal{C})$, is the set of isomorphism classes of invertible objects X, with

$$[X] \cdot [Y] = [X \otimes Y]$$

$$[M]^{-1} = [\text{Hom}_{\mathcal{C}}(X, 1)]$$

Example

We have that $\text{Pic}(R) = \text{Pic}(\text{Mod}(R))$.

However, we had more interesting structure in $\text{Pic}(\text{Ch}(R))$ since we could shift the unit $R[0]$ up or down.

"Definition"

A symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ is called **stable** *if it also has a suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ that is an equivalence of categories.*
In addition, Σ should play nicely with the tensor product. That is, $\Sigma(A \otimes B) \cong \Sigma A \otimes B$.

Warning: This definition is only right when using ∞-categories. Alternatively, we can make a similar definiton using triangulated categories.

Example

The following categories are stable symmetric monoidal:

(a) $(D(R), \hat{\otimes}_R, R[0], -[1])$ for R a commutative ring.

(b) $(\mathrm{Sp}, \wedge, \mathbb{S}, \Sigma)$

(c) $(\mathrm{Mod}(R), \wedge_R, R, \Sigma)$ for R a commutative ring spectrum.

(d) $(L_E(\mathrm{Sp}), L_E(- \wedge -), L_E\mathbb{S}, \Sigma)$ for a spectrum E. In particular, $E = E(n)$ or $K(n)$.

(e) $(\mathrm{StMod}(kG), \otimes_k, k, \Omega^{-1})$ for G a p-group and k a field of characteristic p.

Proposition

Suppose that $(\mathcal{C}, \otimes, 1, \Sigma)$ is a stable symmetric monoidal category. Then one has a natural map

$$\mathbb{Z} \hookrightarrow \mathrm{Pic}(\mathcal{C})$$

$$n \mapsto \Sigma^n 1$$

Theorem (Hopkins-Mahowald-Sadofsky)

$\text{Pic}(\text{Sp}) \cong \mathbb{Z}$

Proof.

Since $X \in \text{Pic}(\text{Sp})$, it is dualizable and therefore finite. We can then assume X is connected.
Then look at the homology of X with field coefficients for all fields and use the Künneth Theorem.
We can then deduce $H_*(X) \cong H_0(X) \cong \mathbb{Z}$ and hence $X \simeq \mathbb{S}$ by the stable Hurewicz and Whitehead theorem. □

Definition

A (commutative) ring spectrum R is a (commutative) ring object in the category of spectra. That is, it has a multiplication that is unital and associative (and commutative).

Example

The following are examples of commutative ring spectra:

(a) $\mathbb{S}$

(b) Given a discrete ring R, we can form the Eilenberg-Maclane spectrum HR. Note that $\pi_*(HR) = R$, viewed as a graded ring concentrated in degree 0.

(c) KU, KO, MU, $E(n)$.

Proposition (Baker-Richter)

We have a monomorphism

$$\Phi : \mathrm{Pic}(\pi_*(R)) \hookrightarrow \mathrm{Pic}(R)$$

Proof.

Given M_*, we build M as a homotopy colimit of free R modules, and use the Künneth Spectral Sequence to check M is a Picard group element.

$$E^2_{p,q} = \mathrm{Tor}^{R_*}_{p,q}(M_*, N_*) \Rightarrow \pi_{p+q}(M \wedge_R N)$$

□

Definition

When $\Phi : \text{Pic}(\pi_*(R)) \to \text{Pic}(R)$ is an isomorphism, then we say that $\text{Pic}(R)$ is **algebraic**.

Theorem (Baker-Richter)

For a connective commutative ring spectrum R, $\text{Pic}(R)$ is algebraic.

Theorem (Baker-Richter)

For a weakly even periodic E_∞ ring spectrum with $\pi_0(R)$ regular Noetherian, $\text{Pic}(R)$ is algebraic.

“Theorem” (Hopkins)

For the spectra $K(n)$ and $E(n)$ at some fixed prime p, the Picard groups $\mathrm{Pic}(L_{E(n)}(\mathrm{Sp}))$ and $\mathrm{Pic}(L_{K(n)}(\mathrm{Sp}))$ are extremely interesting.

Theorem (Hovey-Sadofsky, Kamiya-Shimomura)

$X \in \mathrm{Pic}(L_{E(n)}(\mathrm{Sp})$ if and only if there is an isomorphism $E(n)_*(X) \cong E(n)_*$ as $E(n)_*E(n)$ comodules.

Example

For $n = 1$, $p = 2$, $\mathrm{Pic}(L_{E(n)}(\mathrm{Sp})) \cong \mathbb{Z} \oplus \mathbb{Z}/2$.

Definition

The $E(n)$-based Adams spectral sequence, which takes in input a spectrum X and has E_2 page:

$$E_2^{s,t}(X) = \mathrm{Ext}^s_{E(n)_*E(n)}(E(n)_*, E(n)_t(X)) \Rightarrow \pi_{s+t}(L_nX)$$

and differential (for $r \geq 2$)

$$d_r : E_2^{s,t} \to E_r^{s+r,t+r-1}$$

Theorem (Mathew-Stojanoska)

If $f : R \to S$ is a faithful G-Galois extension of ring spectra, then we have an equivalence of ∞-categories

$$\mathrm{Mod}(R) \to \mathrm{Mod}(S)^{hG}$$

Corollary

The homotopy fixed point spectral sequence, which takes in input the spectrum $pic(S)$ and has E_2 page:

$$H^s(G; \pi_t(pic(S)) \Rightarrow \pi_{t-s}(pic(S)^{hG})$$

whose abutment for $t = s$ is $\mathrm{Pic}(R)$.

Thanks for listening!