What is Algebraic Topology?

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An exercise

How would classify the following shapes?

Motivation

- Broadly speaking, topology is a generalization of the study of shapes.
- These shapes can be abstract and hard to visualize. So we study them though algebraic tools called invariants.
- In particular, algebraic topologists are interested answering the following question: When are two topological spaces are the same or different?

What is a topological space?

- Abstract definition: A set X with a family of subsets τ satisfying certain axioms (called a topology on X). The elements of τ are the open sets.
  1. The empty set and X belong in τ.
  2. Any union of members in τ belongs in τ.
  3. The intersection of a finite number of members in τ belongs in τ.
- Most things are topological spaces.

Example (Euclidean space)

Euclidean space is $\mathbb{R}^n$. A subset U is open if for all $x \in U$, there exists an $\varepsilon > 0$ such that

$$\{y \in U \mid ||x - y|| < \varepsilon\} \subseteq U$$

Example (Disks)

As a subset of $\mathbb{R}^n$, the disk

$$D^n := \{y \in \mathbb{R}^n \mid ||y|| \leq 1\}$$

inherits a topology.

Example (Spheres)

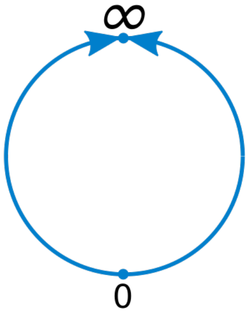
As a subset of $\mathbb{R}^n$, the sphere

$$S^{n-1} := \{y \in \mathbb{R}^n \mid ||y|| = 1\}$$

inherits a topology.

Example (Spheres)

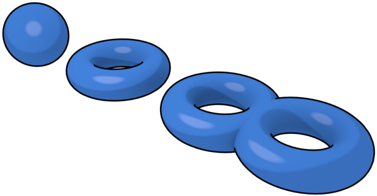
Alternatively, an n-**sphere** is the one-point compactification of $\mathbb{R}^n$. We write it as S^n.



Source: Wikipedia

Example (Surfaces)

A **surface** is a topological space that locally looks like $\mathbb{R}^2$.

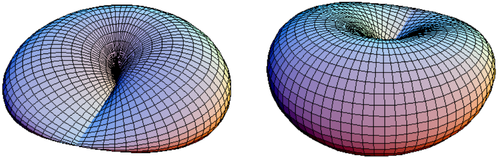


Source: laerne.github.io

Example (Real projective plane)

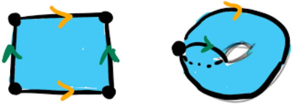
Let $\mathbb{R}P^n$ denote the topological space of lines in R^{n+1} passing through the origin.

This turns out to be an n-manifold, which locally looks like $\mathbb{R}^n$.



A visualization of the real projective plane $\mathbb{R}P^2$. Source: Wikipedia.

Building Topological Spaces

- 2-sphere: 
- 2-sphere: 
- 2-Torus: 
- $\mathbb{R}P^2$:

Building CW complexes

- Abstract toplogical spaces are sometimes hard to get a handle on, so we would like to model them with combinatorial objects, called CW complexes.
- To build a CW complex, you start with a set of points, which is called the 0-skeleton.
- Next, you glue in 1-cells (copies of D^1) to the 0-skeleton, such that the boundary of each D^1 is in the boundary. This forms the 1-skeleton.
- You repeat this process, gluing in n-cells (copies of D^n) such that the boundary of each D^n lies inside the $(n-1)$-skeleton.

Putting CW structures on topological spaces

Theorem (CW approximation theorem)

For every topological space X, there is a CW complex Z and a weak homotopy equivalence $Z \to X$.

How do you relate two topological spaces?

The relationship should preserve the topology, hence we will only consider continuous maps.

Definition (continuous maps)

A map $f : X \to Y$ of topological spaces is **continuous** if for every open set in $U \subseteq Y$, then $f^{-1}(U) \subseteq X$ is open.

Exercise: Compare this to the epsilon-delta definition of continuity for $\mathbb{R}^n$!

When are two topological spaces the same?

A naïve guess: Two spaces X and Y are the same if there exists a map $f : X \to Y$ and a map $g : Y \to X$ such that $f \circ g = id_Y$.

However, we almost never have strict equality. So we must choose a perspective of equality to work with.

- **Homeomorphism**.
- **Homotopy equivalence**.
- **Weak homotopy equivalence**.

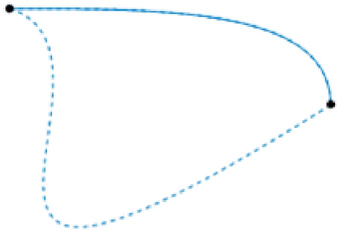
Definition (homeomorphism)

A map $f : X \to Y$ is a **homeomorphism** if f is a bijection such that the inverse $g : Y \to X$ is continuous.

Source: Wikipedia

The coffee cup and donut are homeomorphic.

Homotopy



A homotopy of paths. Source: Wikipedia

Homotopy

Definition (homotopy of maps)

A **homotopy** between two continuous maps $f, g : X \to Y$ is a continuous function $H : X \times [0, 1] \to Y$ such that for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. We write $f \simeq g$.

Proposition

Homotopy defines an equivalence relation on maps from $X \to Y$.

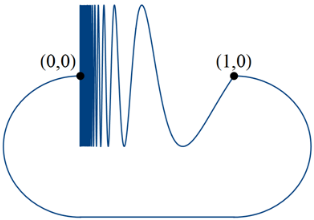
Definition (homotopy equivalence)

A map $f : X \to Y$ is a **homotopy equivalence** if f has a continuous homotopy inverse $g : Y \to X$.
That is, there exists a continuous map $g : Y \to X$ such that $f \circ g \simeq Id_Y$ and $g \circ f \simeq Id_X$.

The annulus is homotopy equivalent, but not homeomorphic, to the point.

Definition (weak homotopy equivalence)

A map $f : X \to Y$ is a **weak homotopy equivalence** if f induces bijections on π_0 and isomorphisms on all homotopy groups.



Source: Math Stackexchange

The Warsaw circle is weakly homotopy equivalent, but not homotopy equivalent, to the point.

Comparison of perspectives

Proposition

Homeomorphism $\Rightarrow$ *Homotopy equivalence* $\Rightarrow$ *Weak homotopy equivalence.*

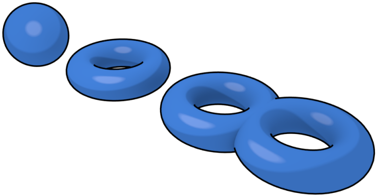
When can we go the other way?

Theorem (Whitehead's theorem)

If $f : X \to Y$ *is a weak homotopy equivalence of CW complexes, then* f *is a homotopy equivalence.*

When are they different?

- It's somehow hard to determine whether or not two spaces are the same. It's much easier to tell spaces apart using tools called **invariants**. These invariants depend on your choice of perspective.



Source: laerne.github.io

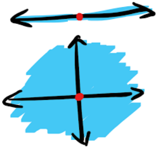
Connectedness

Definition (Connectedness)

A space is **connected** if it cannot be written as the disjoint union of two open sets.

Example

$\mathbb{R} - \{0\}$ is not connected, but $\mathbb{R}^n - \{0\}$ is for $n \geq 2$.



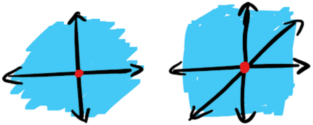
Simple-connectedness

Definition (Simple-connectedness)

A space X is **simply connected** if it is path connected and any loop in X can be contracted to a point.

Example

$\mathbb{R}^2 - \{0\}$ is not simply-connected, but $\mathbb{R}^n - \{0\}$ is for $n \geq 3$.



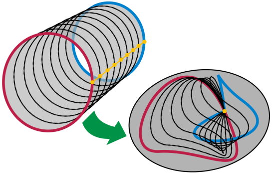
- Connectedness and simple-connectedness are a manifestation of counting the number of 0 and 1-dimensional "holes" in a topological space.
- We can generalize this notion to an algebraic invariant called **homology**.
- This is how we can tell $\mathbb{R}^n$ is not **homeomorphic** to $\mathbb{R}^m$ for $n \neq m$.
- It is much easier to calculate things algebraically, rather than rely on geometry.
- Some other useful invariants are **cohomology** and **homotopy groups**.

Fundamental Group

Let us now assume that X is path-connected.

Proposition

The set of loops on X with a fixed base point up to homotopy form a group, where the multiplication is concatenation.

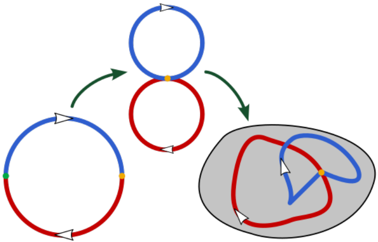


Source: Wikipedia

Fundamental Group

Proposition

The set of homotopy classes of based continuous maps $f : S^1 \to X$ form a group, denoted $\pi_1(X)$.



Source: Wikipedia

Example

If X is contractible, $\pi_1(X) = 0$.

Example

$\pi_1(S^1) \cong \mathbb{Z}$.

This is a foundational result in algebraic topology.

Question: Given a group G, is it possible to build a space X such that $\pi_1(X) \cong G$?

Higher homotopy groups

Proposition

The set of homotopy classes of continuous based maps $f : S^n \to X$ form a group, denoted $\pi_n(X)$

There are lots of calculational tools:

- Long exact sequence of a fibration
- **Spectral sequences**
- Hurewicz theorem
- Blakers-Massey theorem

Higher homotopy groups of spheres

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2 \times \mathbb{Z}/84$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2 \times \mathbb{Z}/84$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/3 \times \mathbb{Z}/24$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2 \times \mathbb{Z}/12 \times \mathbb{Z}/120$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/30$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/60$	$\mathbb{Z}/2 \times \mathbb{Z}/24$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/120$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$
S^9	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0
S^{10}	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0
S^{11}	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$
S^{12}	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
S^{13}	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$
S^{14}	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$

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S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2 \times \mathbb{Z}/84$
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S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/3 \times \mathbb{Z}/24$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2 \times \mathbb{Z}/12 \times \mathbb{Z}/120$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/30$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/60$	$\mathbb{Z}/2 \times \mathbb{Z}/24$
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S^{10}	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0
S^{11}	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$
S^{12}	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
S^{13}	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$
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S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2 \times \mathbb{Z}/84$
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S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/60$	$\mathbb{Z}/2 \times \mathbb{Z}/24$
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S^{10}	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0
S^{11}	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$
S^{12}	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
S^{13}	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$
S^{14}	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$

Higher homotopy groups of spheres

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S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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S^{12}	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
S^{13}	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$
S^{14}	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$

Freudenthal Suspension Theorem

- This is not a coincidence!

Theorem (Corollary of Freudenthal Suspension Theorem)

For $n \geq k + 2$, there is an isomorphism

$$\pi_{k+n}(S^n) \cong \pi_{k+n+1}(S^{n+1})$$

The general theorem says that for fixed k, there is stabilization for highly-connected spaces. We can make spaces highly connected via suspension.

Stable homotopy theory

Definition (stable homotopy groups of spheres)

The k-th stable homotopy group of spheres, $\pi_k^S(S)$, is $\pi_{k+n}(S^n)$ for $n \geq k+2$.

- This is an algebraic phenomenon, and one might wonder if there is a corresponding topological/geometric concept.
- Recall that homotopy groups of X are homotopy classes of maps from $S^n \to X$. Is there a corresponding notion for stable homotopy groups?
- The answer is **yes**!
- This leads to the notion of **spectra**, which is the stable version of a space, and to stable homotopy theory.

Summary

- We would like to understand when two topological spaces are the same or different. This depends on our choice of perspective.
- In particular, we would like to compute invariants that can help us answer this question. We use geometric, combinatorial, and algebraic tools to do so.
- Studying these invariants often leads to fascinating new patterns, which in turn brings us new geometric insights like stable phenomena.