SPECTRAL SEQUENCES TRAINING MONTAGE EXERCISES

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Abstract. These are exercises designed to accompany the 2020 Summer Minicourse "Spectral Sequence Training Montage", led by Arun Debray and Richard Wong. Spectral sequences covered included the Serre SS, the homotopy fixed-point SS, the Atiyah-Hirzebruch SS, the Tate SS, and the Adams SS. Minicourse materials can be found at https://web.ma.utexas.edu/SMC/2020/Resources.html.

Instructor's note: I compiled this list of exercises because there is simply too much material to cover in a one week minicourse. The topics covered in these exercises include: background material; interesting calculations; interesting applications; and questions for your own enlightenment. For each day, I will recommend a subset of exercises that I think are the most important.

1. Monday Exercises

The section on fibrations is background material. I recommend exercises 1.5, 1.6., 1.8, 1.10, 1.13, and 1.17.

1.1. Fibration exercises

Exercise 1.1. Show that if $f : X \to B$ is a Serre fibration with B path-connected, then the fibers over any two points are homotopy equivalent. That is, $f^{-1}(b_1) \simeq f^{-1}(b_2)$.

Exercise 1.2. Show that a Serre fibration $F \to E \to B$ induces a long exact sequence of homotopy groups

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots \to \pi_0(E)$$

Exercise 1.3. Given a short exact sequence of groups $H \to G \to G/H$, show that there is a Serre fibration of classifying spaces $BH \to BG \to BG/H$

Exercise 1.4. Show that the notion of Serre fibration is strictly weaker than the notion of a Hurewicz fibration.

Exercise 1.5. Show that the fibration $G \to EG \to BG$ can be obtained from the path space fibration $\Omega BG \to BG^I \to BG$.

Exercise 1.6. Given a universal cover $\tilde{X} \to X$ with $\pi_1(X) = G$, show that we have a fibration $\tilde{X} \to X \to BG$.

In general, If G acts on a space X such that the quotient map $X \to X/G$ is a covering space, show that we have a fibration $X \to X/G \to BG$.

1.2. Spectral Sequence Computations

Exercise 1.7. Given a universal cover $\tilde{X} \to X$ with $\pi_1(X) = G$ (with G finite), use the Serre spectral sequence to show that there is an isomorphism $H^*(X;\mathbb{Q}) \to (H^*(\tilde{X};\mathbb{Q}))^G$.

How can this statement be generalized? For example, how necessary is the coefficient ring $\mathbb{Q}$?

Exercise 1.8. Show that if $F \to E \to B$ is a Serre fibration with $\pi_1(B)$ acting trivially, and we take coefficients $A = k$ for some field k, then the Serre spectral sequence takes the form

$$E_2^{s,t} = H^p(B;k) \otimes H^q(F;k) \Rightarrow H^{p+q}(E;k)$$

Exercise 1.9. Play around with the Serre spectral sequence for the Hopf fibration $S^1 \to S^3 \to S^2$.

Exercise 1.10. Play around with the Serre spectral sequence for the fibration $U(n-1) \to U(n) \to S^{2n-1}$.

Exercise 1.11. Play around with the Serre spectral sequence for the fibration $SO(n) \to SO(n+1) \to S^n$.

Exercise 1.12. Let $V_2(\mathbb{R}^{n+1})$ be the space of orthogonal pairs of vectors in $\mathbb{R}^{n+1}$.

(1) Show we have a Serre fibration $S^{n-1} \to V_2(\mathbb{R}^{n+1}) \to S^n$
(2) Compute $H^*(V_2(\mathbb{R}^{n+1}))$.

Exercise 1.13. Compute the cup product structure on $H^*(\Omega S^n)$ using the path space fibration $\Omega S^n \to (S^n)^I \to S^n$.

Exercise 1.14. Compare the spectral sequence for the fibration $S^2 \to S^2 \times S^2 \to S^2$ with the fibration $S^2 \to X \to S^2$, where X is built by taking two mapping cylinders of the Hopf map $S^3 \to S^2$, and gluing them together along the identity on S^3.

Show that $H^*(S^2 \times S^2)$ and $H^*(X)$ have different ring structures.

Exercise 1.15. Prove (recover) the Gysin sequence.

Theorem (The Gysin Sequence)**.** Let $S^n \to E \to B$ be a Serre fibration with B simply connected and $n \geq 1$. There exists a long exact sequence

$$\cdots \to H^k(B) \to H^k(X) \to H^{k-n}(B) \to H^{k+1}(B) \to \cdots$$

Exercise 1.16. Prove (recover) the Wang sequence.

Theorem (The Wang Sequence)**.** Let $F \to X \to S^n$ be a Serre fibration with B simply connected and $n \geq 1$. There exists a long exact sequence

$$\cdots \to H^{k-1}(F) \to H^{k-n}(F) \to H^k(X) \to H^k(F) \to \cdots$$

Exercise 1.17. Prove (recover) this Hurewicz isomorphism using the path fibration

$$\Omega(X) \to PX \to X$$

Theorem (Hurewicz)**.** Let X be an $(n-1)$-connected space, with $n \geq 2$. Then $\tilde{H}_i(X) = 0$ for $i \leq n-1$, and we have the Hurewicz isomorphism

$$\pi_n(X) \cong H_n(X)$$

Exercise 1.18. Prove (recover) the Leray-Hirsch Theorem.

Theorem (Leray-Hirsch)**.** Let $F \to E \to B$ be a fiber bundle such that F is of finite type. That is, that $H^p(F;\mathbb{Q})$ is finite dimensional for all p.

Furthermore, assume that the inclusion $i : F \to E$ induces a surjection

$$i^* : H^*(E;\mathbb{Q}) \to H^*(F;\mathbb{Q})$$

Then we have an isomorphism of $H^*(B;\mathbb{Q})$-modules

$$H^*(F;\mathbb{Q}) \otimes_{\mathbb{Q}} H^*(B;\mathbb{Q}) \cong H^*(E;\mathbb{Q})$$

Exercise 1.19. How can the Leray-Hirsch theorem above be generalized? In particular, how necessary is the coefficient ring $\mathbb{Q}$?

1.3. **For your enlightenment**

Exercise 1.20. Show that there is a relationship between the bigraded chain complex

$$\cdots \to H^*(E_{s-1}, E_s) \xrightarrow{d} H^*(E_s, E_{s-1}) \xrightarrow{d} H^*(E_{s+1}, E_s) \to \cdots$$

and $H^*(B)$ and $H^*(F)$.

Namely, that there is an isomorphism

$$E_1^{s,t} \cong C^s(B; H^t(F))$$

where $C^*(B; H^t(F))$ is the cellular cochain complex for B with coefficients in $H^t(F)$.

Exercise 1.21. What was special about the Serre filtration on X? Can you construct exact couples using a different filtration? Can you construct a spectral sequence using a different filtration?

Exercise 1.22. What was special about using cohomology? Can you construct a homological Serre spectral sequence?

Can you construct a spectral sequence using a generalized cohomology theory?

2. Tuesday Exercises

There is subsection on groups acting freely on spheres, which is an interesting application of group cohomology. I recommend exercises 2.1, 2.2, 2.3, 2.10, 2.13, and 2.14.

2.1. Group Cohomology

Exercise 2.1. Show that there is an isomorphism

$$H^*(G;M) \cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, M)$$

This gives us an algebraic way to compute group cohomology.

Exercise 2.2. Let M be a $\mathbb{Z}G$-module. Show that $H^0(G;M) = M^G$, the G-fixed points of M.

Exercise 2.3. Compute the LHS spectral sequence with coefficients in a field of characteristic p for the fibration $B(\mathbb{Z}/2)^2 \to BA_4 \to B\mathbb{Z}/3$

Exercise 2.4. Compute $H^*(BD_8;\mathbb{F}_2)$ using the LHS spectral sequence.

Exercise 2.5. Compute $H^*(BQ_8;\mathbb{F}_2)$ using the LHS spectral sequence.

Exercise 2.6. Show that for G a finite group, and a faithful unitary representation $\Phi : G \to U(n)$ with Chern classes $c_i(\Phi)$, then

$$|G| \Big| \prod_{i=1}^{n} exp(c_i(\Phi))$$

Hint: Consider the fibration $G \to U(n) \to U(n)/G$.

Exercise 2.7. Let G be a finite group of order n. Show that $n \cdot H^i(G;M) = 0$ for any G-module M. That is, that group cohomology is always $|G|$-torsion.

Hint: consider the restriction and transfer maps $\mathrm{res}^G_H : H^*(G;M) \to H^*(H;M)$ and $\mathrm{tr}^G_H : H^*(H;M) \to H^*(G;M)$. Show that the composite $\mathrm{tr}^G_H \circ \mathrm{res}^G_H$ is multiplication by the index $|G : H|$.

2.2. Groups acting on Spheres

Exercise 2.8. Show that $\mathbb{Z}/n$ are the only finite groups that act freely on S^1.

Exercise 2.9. Show that if n is even, then the only non-trivial finite group that can act freely on S^n is $\mathbb{Z}/2$.

Exercise 2.10. A finite group G is **periodic** of period $k > 0$ if $H^i(G;\mathbb{Z}) \cong H^{i+k}(G;\mathbb{Z})$ for all $i \geq 1$, where $\mathbb{Z}$ has trivial G action.

Show that if G acts freely on S^n, then G is periodic of period $n+1$.

Exercise 2.11. Show that $\mathbb{Z}/p \times \mathbb{Z}/p$ does not act freely on S^n:

Exercise 2.12. Show that not every periodic group with period 4 acts freely on S^3. (Consider $G = S_3$).

2.3. The HFPSS

Exercise 2.13. Consider the fiber sequence of spaces

$$G/N \to BN \to BG$$

, and the morphism of ring spectra $k^{hG} \to k^{hN}$ obtained by taking cochains with Hk-valued coefficients.

Compare this HFPSS with the Serre spectal sequence.

Exercise 2.14. Consider the fiber sequence $S^1 \to B\mathbb{Z}/2 \to BS^1$. Taking cochains with $\mathbb{F}_2$-valued coefficients, we obtain a morphism of ring spectra $H\mathbb{F}_2^{hS^1} \to H\mathbb{F}_2^{h\mathbb{Z}/2}$.

Compute the HFPSS for the S^1-action on $H\mathbb{F}_2^{h\mathbb{Z}/2}$

Exercise 2.15. Let p be an odd prime. Consider the fiber sequence $S^1 \to B\mathbb{Z}/p \to BS^1$. Taking cochains with $\mathbb{F}_p$-valued coefficients, we obtain a morphism of ring spectra $H\mathbb{F}_p^{hS^1} \to H\mathbb{F}_p^{h\mathbb{Z}/p}$.

Compute the HFPSS for the S^1-action on $H\mathbb{F}_p^{h\mathbb{Z}/p}$

2.4. **For your enlightenment**

Exercise 2.16. Construct BG for a topological group G. Is $BG \simeq K(G,1)$?

Exercise 2.17. If G is a Lie group, what conditions on a subgroup H give a fibration $BH \to BG \to BG/H$?

Exercise 2.18. Generalize the idea of cohomology with local coefficients for a space X with universal cover $\tilde{X}$.

Exercise 2.19. Show that the cohomology of X with local coefficients in $\mathbb{Z}[\pi(X)]$ is isomorphic to the cohomology of the universal cover of X, $\tilde{X}$. That is,

$$H_n(X;\mathbb{Z}[\pi(X)]) \cong H_n(\tilde{X})$$

Exercise 2.20. Dual to the notion of group cohomology, there is a notion of group homology.

Show that there is an isomorphism

$$H_*(G;M) \cong \operatorname{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, M)$$

This gives us an algebraic way to compute group homology.

Exercise 2.21. Let M be a $\mathbb{Z}G$-module. Show that $H_0(G;M) = M_G$, the G-orbits or coinvariants of M. In other words, M_G is the quotient of M by the submodule generated by elements of the form $g \cdot m - m$.

Exercise 2.22. Dual to the notion of homotopy fixed points, there is a notion of homotopy orbits.

Construct the homotopy orbit spectral sequence.

Exercise 2.23. Show that for R a ring, and G acting trivially on the Eilenberg-Maclane spectrum HR, there is an isomorphism

$$\pi_*((HR)_{hG}) \cong H_*(G;R)$$

3. Wednesday Exercises

These were compiled by Arun, see the minicourse website.

4. Thursday Exercises

4.1. **Trickier Serre spectral sequence questions**

Exercise 4.1. Similarly to the example given in lecture today, investigate $H^*(G;\mathbb{Z}_{w_1(\rho)})$ in the first few degrees using the multiplicative structure when combined with $H^*(G;\mathbb{Z})$, for the following groups.

1. $G = \mathrm{O}(2)$, and ρ is the standard two-dimensional real representation. The extension is $1 \to \mathrm{SO}(2) \to \mathrm{O}(2) \to \mathbb{Z}/2 \to 1$.
2. $G = D_{2n}$, and ρ is the two-dimensional real representation of rotations and reflections. The extension is $1 \to C_n \to D_{2n} \to \mathbb{Z}/2 \to 1$. Notes:
   - These spectral sequences are compatible for different n, and via $D_{2n} \to \mathrm{O}(2)$, with the spectral sequence in the previous part of the problem.
   - The spectral sequences will depend on the parity of n, and probably also on n mod 4.
   - The integral cohomology ring of D_{2n} is given here: https://math.stackexchange.com/questions/1294806.

Exercise 4.2. Can you prove that $H^*(\mathbb{Z}/2;R) \cong \mathbb{Z}[e]/(2e)$ with $|e| = (1,-)$?

Exercise 4.3. Challenge question: let's compute the Stiefel-Whitney classes of the *Wu manifold* $W := \mathrm{SU}(3)/\mathrm{SO}(3)$. This will prove that W is not null-bordant; in fact, it is the generator of $\Omega_5^{\mathrm{O}} \cong \mathbb{Z}/2$, representing the lowest-degree element that can't be built using real projective spaces.

1. This part isn't as hard: use the Serre spectral sequence to show that $H^*(W;\mathbb{Z}/2) \cong \mathbb{Z}/2[z_2, z_3]/(z_2^2, z_3^2)$, with $|z_i| = i$. Notes: there is a diffeomorphism $\mathrm{SO}(3) \cong \mathbb{RP}^3$, and $H^*(\mathrm{SU}(3);\mathbb{Z}/2) \cong \mathbb{Z}/2[x_3, x_5]/(x_3^2, x_5^2)$ with $|x_i| = i$. (Can you prove this with another Serre spectral sequence argument?)
2. Now, use the method of Borel-Hirzebruch, as outlined in https://math.stackexchange.com/questions/581401, to compute the Stiefel-Whitney classes of W.

4.2. **Tate Cohomology / SS Exercises**

Exercise 4.4. Let $G \cong \mathbb{Z}/p$. Compute group homology $H_*(G; \mathbb{F}_p)$ for the trivial G action.

Exercise 4.5. Let $G \cong \mathbb{Z}/p$. Compute the norm map for the trivial G action on $\mathbb{F}_p$.

Exercise 4.6. Let $G \cong \mathbb{Z}/p$. Compute the Tate cohomology $\widehat{H}^*(G; \mathbb{F}_p)$ for the trivial G action.

Exercise 4.7. Let $G \cong \mathbb{Z}/n$ be a finite cyclic group. Compute group homology $H_*(G; \mathbb{Z})$ for the trivial G action.

Exercise 4.8. Let $G \cong \mathbb{Z}/n$ be a finite cyclic group. Compute the norm map for the trivial G action on $\mathbb{Z}$.

Exercise 4.9. Let $G \cong \mathbb{Z}/n$ be a finite cyclic group. Compute the Tate cohomology $\widehat{H}^*(G; \mathbb{Z})$ for the trivial G action.

Show that for all $n \in \mathbb{Z}$, there is an isomorphism

$$\widehat{H}^n(G; \mathbb{Z}) \cong \widehat{H}^{n+2}(G; \mathbb{Z})$$

Exercise 4.10. Let G be a finite group such that $p \mid |G|$. Show that the map of ring spectra induced by the fiber sequence $G \to EG \to BG$

$$(H\mathbb{F}_p)^{hG} \to H\mathbb{F}_p$$

is a **non-faithful** Galois extension of ring spectra.

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