Spectral Sequence Training Montage, Day 1

Arun Debray and Richard Wong

Summer Minicourses 2020

Slides, exercises, and video recordings can be found at
https://web.ma.utexas.edu/SMC/2020/Resources.html

Problem Session

There will be an interactive problem session every day, and participation is strongly encouraged.

We are using the free (sign-up required) A Web Whiteboard website. The link will be posted in the chat, as well as on the slack channel.

Future problem sessions will be from 1-1:30pm and 2:30-3pm CDT.

Motivation

Let $\tilde{X} \to X$ be a universal cover of X, with $\pi_1(X) = G$.

What can one say about the relationship between $H^*(\tilde{X}; \mathbb{Q})$ and $H^*(X; \mathbb{Q})$?

Theorem

There is an isomorphism $H^*(X; \mathbb{Q}) \to (H^*(\tilde{X}; \mathbb{Q}))^G$

Proof.

The sketch involves looking at the cellular cochain complex for X, lifting it to a cellular cochain complex for $\tilde{X}$ that is compatible with the G action... □

How can we generalize this theorem?

Definition

Let $F \to E \to B$ be a Serre fibration with B path-connected. We then have the **Serre spectral sequence for cohomology** (with coefficients A):

$$E_2^{s,t} = H^p(B; H^q(F; A)) \Rightarrow H^{p+q}(E; A)$$

with differential

$$d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$$

The key property of covering spaces that we use is the **homotopy lifting property**:

Definition (Homotopy lifting property)

A map $f : E \to B$ has the homotopy lifting property with respect to a space X if for any homotopy $g_t : X \times I \to B$ and any map $\tilde{g}_0 : X \to E$, there exists a map $\tilde{g}_t : X \times I \to E$ lifting the homotopy g_t.

$$\begin{array}{ccc} X & \xrightarrow{\tilde{g}_0} & E \\ {\scriptstyle X\times\{0\}}\downarrow & \exists \tilde{g}_t \nearrow & \downarrow{\scriptstyle f} \\ X \times I & \xrightarrow[g_t]{} & B \end{array}$$

Definition

A map $f : E \to B$ is called a (Hurewicz) fibration if it has the homotopy lifting property for all spaces X.

Definition

A map $f : E \to B$ is called a Serre fibration if it has the homotopy lifting property for all disks (or equivalently, CW complexes).

We will only consider fibrations with B path-connected. This implies that the fibers $F = f^{-1}(b)$ are all homotopy equivalent, and so we write fibrations in the form

$$F \to E \to B$$

Example

The universal cover $\tilde{X} \to X$ is a fibration with fiber $F = \pi_1(X)$.

Example

The projection map $X \times Y \xrightarrow{p_1} X$ is a fibration with fiber Y.

Example

The Hopf map $S^1 \to S^3 \to S^2$ is a fibration.

Example

For any based space $(X, *)$, there is the path space fibration

$$\Omega X \to X^I \to X$$

Where X^I is the space of continuous maps $f : I \to X$ with $f(0) = *$. Note that $X^I \simeq *$.

Example

For G abelian, and $n \geq 1$, we have fibrations

$$K(G, n) \to * \to K(G, n+1)$$

Example

For G a group, we have the fibration $G \to EG \to BG$

Given a Serre fibration $F \to E \to B$, how can we relate the cohomology of E to the cohomology of B?

Remark

Note that by putting a CW-structure on B, we have a filtration

$$B_0 \subseteq B_1 \subseteq \cdots \subseteq B$$

This lifts to the Serre filtration on E:

$$E_0 = p^{-1}(B_0) \subseteq E_1 = p^{-1}(B_1) \subseteq \cdots \subseteq E$$

Using the Serre filtration, we can assemble the long exact sequences in relative cohomology:

$$\begin{array}{ccccccccccc}
 & & \downarrow & & & & \downarrow & & & & \downarrow & \\
\longrightarrow & & H^{n-1}(E_s) & \longrightarrow & H^n(E_{s+1}, E_s) & \longrightarrow & H^n(E_{s+1}) & \rightarrow & H^{n+1}(E_{s+2}, E_{s+1}) & \rightarrow & H^{n+1}(E_{s+2}) & \rightarrow \\
 & & \downarrow & & & & \downarrow & & & & \downarrow & \\
\rightarrow & & H^{n-1}(E_{s-1}) & \longrightarrow & H^n(E_s, E_{s-1}) & \longrightarrow & H^n(E_s) & \longrightarrow & H^{n+1}(E_{s+1}, E_s) & \longrightarrow & H^{n+1}(E_{s+1}) & \rightarrow \\
 & & \downarrow & & & & \downarrow & & & & \downarrow & \\
\rightarrow & & H^{n-1}(E_{s-2}) & \rightarrow & H^n(E_{s-1}, E_{s-2}) & \rightarrow & H^n(E_{s-1}) & \longrightarrow & H^{n+1}(E_s, E_{s-1}) & \longrightarrow & H^{n+1}(E_s) & \longrightarrow \\
 & & \downarrow & & & & \downarrow & & & & \downarrow &
\end{array}$$

We obtain a long exact sequence

$$\cdots \to H^n(E_{s+1}) \xrightarrow{i} H^n(E_s) \xrightarrow{j} H^{n+1}(E_{s+1}, E_s) \xrightarrow{k} H^{n+1}(E_{s+1}) \to \cdots$$

We can rewrite this long exact sequence as an unrolled **exact couple**:

$$
\begin{array}{ccccccccc}
H^*(E) \to \cdots \to & H^*(E_{s+1}) & \xrightarrow{i} & H^*(E_s) & \xrightarrow{i} & H^*(E_{s-1}) & \longrightarrow \cdots \\
& & {\scriptstyle k}\nwarrow & \downarrow {\scriptstyle j} & {\scriptstyle k}\nwarrow & \downarrow {\scriptstyle j} & \\
& & & H^*(E_{s+1}, E_s) & & H^*(E_s, E_{s-1}) &
\end{array}
$$

Remark

Observe that this diagram is not commutative.

Furthermore, since $k \circ j = 0$, *the composite*

$$d := j \circ k : H^*(E_s, E_{s-1}) \to H^*(E_{s+1}, E_s)$$

can be thought of as a chain complex differential, as $d^2 = 0$.

We have a bigraded chain complex

$$\cdots \to H^*(E_{s-1}, E_s) \xrightarrow{d} H^*(E_s, E_{s-1}) \xrightarrow{d} H^*(E_{s+1}, E_s) \to \cdots$$

We call this chain complex the E_1 page of the Serre spectral sequence.

- How does this chain complex relate to $H^*(E)$?
- How does this chain complex relate to $H^*(B)$ and $H^*(F)$?
- What happens if we take the homology of this chain complex?
    - We get another exact couple, and the E_2 page of the Serre spectral sequence.

Definition

Let $F \to E \to B$ be a Serre fibration with B path-connected. We then have the **Serre spectral sequence for cohomology** (with coefficients A):

$$E_2^{s,t} = H^p(B; H^q(F; A)) \Rightarrow H^{p+q}(E; A)$$

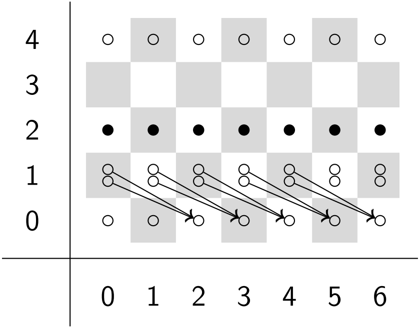
with differential

$$d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$$

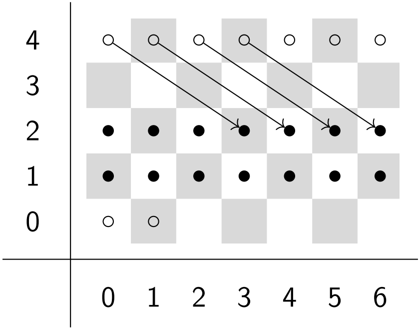
Remark

Some formulations of the Serre spectral sequence require that $\pi_1(B) = 0$, or that $\pi_1(B)$ acts trivially on $H^*(F; A)$.

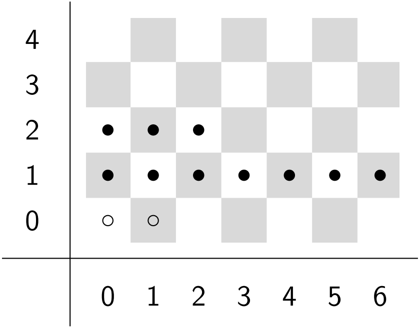
This assumption only exists so that one only needs to consider ordinary cohomology, as opposed to working with cohomology with local coefficients.



An example E_2 page of the Serre Spectral Sequence. $\circ = \mathbb{Z}$, $\bullet = \mathbb{Z}/2$.



An example E_3 page of the Serre Spectral Sequence. $\circ = \mathbb{Z}$, $\bullet = \mathbb{Z}/2$.



An example $E_4 = E_\infty$ page of the Serre Spectral Sequence. $\circ = \mathbb{Z}$, $\bullet = \mathbb{Z}/2$.

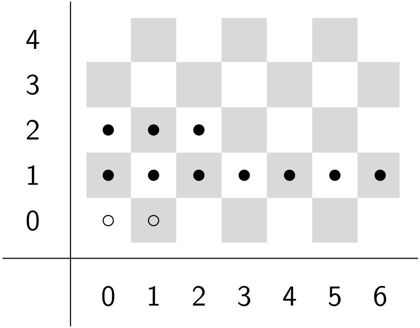
In the Serre spectral sequence, we have that $E_r^{s,t} \cong E_{r+1}^{s,t}$ for sufficiently large r. We call this the E_∞-**page**.

Moreover, the spectral sequence **converges** to $H^*(E; A)$ in the following sense: The E_∞-page is isomorphic to the **associated graded** of $H^*(E)$.

This means that for $F_s^t = \ker(H^t(E) \to H^t(E_{s-1}))$, we have

$$\bigoplus_t E_\infty^{s,t} \cong \bigoplus_t F_s^t / F_s^{t+1}$$

Therefore, we can calculate $H^*(E; A)$ up to group extension. We can sometimes recover the multiplicative structure of $H^*(E; A)$ as well.



An example $E_4 = E_\infty$ page of the Serre Spectral Sequence. $\circ = \mathbb{Z}$, $\bullet = \mathbb{Z}/2$.

Definition

Let $F \to E \to B$ be a Serre fibration with B path-connected. We then have the **Serre spectral sequence for cohomology** (with coefficients A):

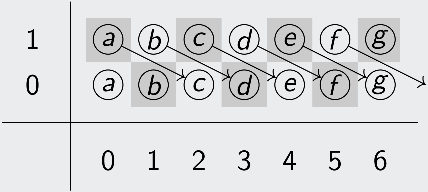
$$E_2^{s,t} = H^p(B; H^q(F; A)) \Rightarrow H^{p+q}(E; A)$$

with differential

$$d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$$

Example

Consider the path space fibration $K(\mathbb{Z},1) \to K(\mathbb{Z},2)^I \to K(\mathbb{Z},2)$
We know that $K(\mathbb{Z},1) \simeq S^1$, and we know $K(\mathbb{Z},2)^I \simeq *$

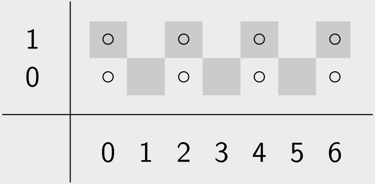


The E_2 page and possible non-trivial differentials

Since $K(\mathbb{Z},2)$ is connected, $a \cong \mathbb{Z}$. Therefore, the d_2 out of $(0,1)$ must be non-trivial, and in fact an isomorphism.

Example

Similarly, since b in $(1,0)$ cannot hit or be hit by a d_2 differential, it must be trivial.



The $E_3 = E_\infty$ page. $\circ = \mathbb{Z}$.

Hence $H^s(K(\mathbb{Z},2);\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & s \text{ even}, \geq 0 \\ 0 & \textit{else} \end{cases}$.

In fact, $K(\mathbb{Z},2) \simeq \mathbb{C}P^\infty$.

Recall that $H^*(E;R)$ has a ring structure if we take coefficients in a ring R. This is compatible with the Serre spectral sequence: Each d_r is a derivation, satisfying

$$d_r(xy) = d_r(x)y + (-1)^{p+q} x d_r(y)$$

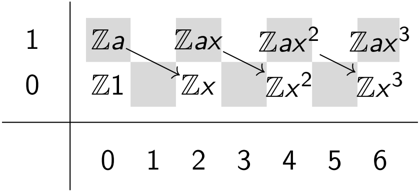
for $x \in E_r^{s,t}$, $y \in E_r^{s',t'}$. This induces a product structure on each E_r, and hence a product stucture on the E_∞-page.
The product structure on E_2 is derived from the multiplication

$$H^s(B; H^t(F;R)) \times H^{s'}(B; H^{t'}(F;R)) \to H^{s+s'}(B; H^{t+t'}(F;R))$$

The multiplication on $H^*(E;R)$ restricts to the associated graded, and is identified with the product on E_∞.

Warning

The ring structure on E_∞ may not determine the ring structure on $H^*(E)$. See the exercises for a counterexample.



The E_2 page for $K(\mathbb{Z},1) \to K(\mathbb{Z},2)^I \to K(\mathbb{Z},2)$.

Since $d_2 : \mathbb{Z}a \to \mathbb{Z}x$ is an isomorphism, we may assume that $d_2(a) = x$. Furthermore,

$$d_2(ax^i) = d_2(a)x^i + d_2(x^i)a = d_2(a)x^i$$

Therefore, $H^*(K(\mathbb{Z},2);\mathbb{Z}) \cong \mathbb{Z}[x]$. In fact, $K(\mathbb{Z},2) \simeq \mathbb{C}P^\infty$.

Problem Session

You can find the exercises at https://web.ma.utexas.edu/SMC/2020/Resources.html.

We are using the free (sign-up required) A Web Whiteboard website. The link will be posted in the chat, as well as on the slack channel.

Future problem sessions will be from 1-1:30pm and 2:30-3pm CDT.