Picard Groups of the Stable Module Category for Quaternion Groups

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I use the computational methods of homotopy theory to study the **modular representation theory** of finite groups G over a field k of characteristic p, where $p \mid |G|$.

Definition

The **group of endo-trivial modules** is the group

$$T(G) := \{M \in \mathrm{Mod}(kG) \mid \mathrm{End}_k(M) \cong k \oplus P\}$$

where k is the trivial kG-module, and P is a projective kG-module.

We can understand this group as the Picard group of the **stable module category** StMod(kG):

$$T(G) \cong \mathrm{Pic}(\mathrm{StMod}(kG))$$

Theorem (van de Meer-W., cf Carlson-Thévenaz)

Let ω denote a cube root of unity.

$$\mathrm{Pic}(\mathrm{StMod}(kQ_8)) \cong \begin{cases} \mathbb{Z}/4 & \text{if } \omega \notin k \\ \mathbb{Z}/4 \oplus \mathbb{Z}/2 & \text{if } \omega \in k \end{cases}$$

Theorem (van de Meer-W., cf Carlson-Thévenaz)

Let $n \geq 4$.

$$\mathrm{Pic}(\mathrm{StMod}(kQ_{2^n})) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$$

Definition

The **Picard group** of a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$, denoted $\text{Pic}(\mathcal{C})$, is the set of isomorphism classes of invertible objects X, with

$$[X] \cdot [Y] = [X \otimes Y]$$

$$[X]^{-1} = [\text{Hom}_{\mathcal{C}}(X, 1)]$$

Example (Hopkins-Mahowald-Sadofsky)

For $(\text{Sp}, \wedge, \mathbb{S}, \Sigma)$ the stable symmetric monoidal category of spectra,

$$\text{Pic}(\text{Sp}) \cong \mathbb{Z}$$

Given a symmetric monoidal ∞-category $\mathcal{C}$, one can do better than the Picard group:

Definition

The **Picard space** $\mathcal{P}\mathrm{ic}(\mathcal{C})$ is the ∞-groupoid of invertible objects in $\mathcal{C}$ and isomorphisms between them.

This is a group-like E_∞-space, and so we equivalently obtain the connective **Picard spectrum** $\mathfrak{pic}(\mathcal{C})$.

Proposition (Mathew-Stojanoska)

The functor $\mathfrak{pic} : \mathrm{Cat}^{\otimes} \to \mathrm{Sp}_{\geq 0}$ *commutes with limits and filtered colimits.*

Example

Let R be an E_∞-ring spectrum. Then Mod(R) is a stable symmetric monoidal ∞-category.

The homotopy groups of $\mathfrak{pic}(R) := \mathfrak{pic}(\mathrm{Mod}(R))$ are given by:

$$\pi_*(\mathfrak{pic}(R)) \cong \begin{cases} \mathrm{Pic}(R) & * = 0 \\ (\pi_0(R))^\times & * = 1 \\ \pi_{*-1}(\mathfrak{gl}_1(R)) \cong \pi_{*-1}(R) & * \geq 2 \end{cases}$$

Galois Descent

Theorem (Mathew-Stojanoska)

*If $f : R \to S$ is a **faithful** G-**Galois extension** of E_∞ ring spectra, then we have an equivalence of ∞-categories*

$$\mathrm{Mod}(R) \cong \mathrm{Mod}(S)^{hG}$$

Corollary

*We have the **homotopy fixed point spectral sequence**, which has input the G action on $\pi_*(pic(S))$:*

$$H^s(G; \pi_t(pic(S)) \Rightarrow \pi_{t-s}(pic(S)^{hG})$$

whose abutment for $t = s$ is $\mathrm{Pic}(R)$.

Definition

The **stable module category** $\mathrm{StMod}(kG)$ has objects kG-modules, and has morphisms

$$\underline{\mathrm{Hom}}_{kG}(M,N) = \mathrm{Hom}_{kG}(M,N)/\mathrm{PHom}_{kG}(M,N)$$

where $\mathrm{PHom}_{kG}(M,N)$ is the linear subspace of maps that factor through a projective module.

Proposition

$\mathrm{StMod}(kG)$ *is a stable symmetric monoidal* ∞*-category.*

From now on, we restrict our attention to the case that G is a finite p-group, so that the following theorem holds:

Theorem (Keller, Mathew, Schwede-Shipley)

There is an equivalence of symmetric monoidal ∞-categories

$$\mathrm{StMod}(kG) \simeq \mathrm{Mod}(k^{tG})$$

Where k^{tG} is an E_∞ ring spectrum called the G-Tate construction.

We will use Galois descent to compute

$$T(G) \cong \mathrm{Pic}(\mathrm{StMod}(kG)) \cong \mathrm{Pic}(k^{tG})$$

Let the spectrum $k^{hG} \simeq F(BG_+, k)$ denote the G-**homotopy fixed points** of k with the trivial action.

Theorem

We have the ***homotopy fixed point spectral sequence****:*

$$E_2^{s,t}(k) = H^s(G; \pi_t(k)) \Rightarrow \pi_{t-s}(k^{hG})$$

and differentials

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$$

Proposition

There is an isomorphism of graded rings

$$\pi_{-*}(k^{hG}) \cong H^*(G; k)$$

There is also $k_{hG} = BG_+ \wedge k$, the G-**homotopy orbits** with the trivial action.

Theorem

*We have the **homotopy orbit spectral sequence**:*

$$E_2^{s,t}(k) = H_s(G; \pi_t(k)) \Rightarrow \pi_{s+t}(k_{hG})$$

Proposition

There is an isomorphism

$$\pi_*(k_{hG}) \cong H_*(G; k)$$

Just like there is a norm map in group cohomology

$$N_G : H_*(G; k) \to H^*(G; k)$$

there is a norm map $N_G : k_{hG} \to k^{hG}$.

And just as one can stitch together group homology and cohomology via the norm map to form Tate cohomology,

$$\widehat{H}^i(G; k) \cong \begin{cases} H^i(G; k) & i \geq 1 \\ \mathrm{coker}(N_G) & i = 0 \\ \ker(N_G) & i = -1 \\ H_{-i-1}(G; k) & i \leq -2 \end{cases}$$

Definition

The G-**Tate construction** is the cofiber of the norm map:

$$k_{hG} \xrightarrow{N_G} k^{hG} \to k^{tG}$$

Theorem

*We have the **Tate spectral sequence**:*

$$E_2^{s,t}(k) = \widehat{H}^s(G; \pi_t(k)) \Rightarrow \pi_{t-s}(k^{tG})$$

Proposition

For G with the trivial action, there is an isomorphism

$$\pi_{-*}(k^{tG}) \cong \widehat{H}^*(G; k)$$

Remark

The multiplication of elements in negative degrees in $\pi_*(k^{tG})$ is the same as the multiplication in $\pi_*(k^{hG})$.

Multiplication by elements in positive degrees is complicated. For example, if $G \cong (\mathbb{Z}/p)^n$ for $n \geq 2$, or if $G \cong D_{2^n}$, then

$$\pi_n(k^{tG}) \cdot \pi_m(k^{tG}) = 0$$

for all $n, m > 0$.

Theorem (Mathew, Schwede-Shipley)

There is an equivalence of symmetric monoidal ∞-categories

$$\mathrm{StMod}(kQ) \simeq \mathrm{Mod}(k^{tQ})$$

where k^{tQ} is an E_∞ ring spectrum called the Q-Tate construction.

Theorem (Mathew-Stojanoska)

If $k^{tQ} \to S$ is a faithful G-Galois extension of E_∞ ring spectra, then we have the HFPSS:

$$H^s(G; \pi_t(pic(S)) \Rightarrow \pi_{t-s}(pic(S)^{hG})$$

whose abutment for $t = s$ is $\mathrm{Pic}(k^{tQ}) \cong \mathrm{Pic}(\mathrm{StMod}(kQ))$.

Definition

A map $f : R \to S$ of E_∞-ring spectra is a G-**Galois extension** if the maps

(i) $i : R \to S^{hG}$

(ii) $h : S \otimes_R S \to F(G_+, S)$

are weak equivalences.

Definition

A G-Galois extension of E_∞-ring spectra $f : R \to S$ is said to be **faithful** if the following property holds:

If M is an R-module such that $S \otimes_R M$ is contractible, then M is contractible.

Proposition (Rognes)

A G-Galois extension of E_∞-ring spectra $f : R \to S$ is faithful if and only if the Tate construction S^{tG} is contractible.

Proposition (van de Meer-W.)

For Q a quaternion group with center $H \cong \mathbb{Z}/2$,

$$k^{hQ} \to k^{h\mathbb{Z}/2} \text{ and } k^{tQ} \to k^{t\mathbb{Z}/2}$$

are faithful Q/H-Galois extensions of ring spectra.

Lemma

$$\pi_*(k^{h\mathbb{Z}/2}) \cong k[t^{-1}] \qquad \pi_*(k^{t\mathbb{Z}/2}) \cong k[t^{\pm 1}]$$

Lemma

Note that $Q_8/H \cong (\mathbb{Z}/2)^2$.

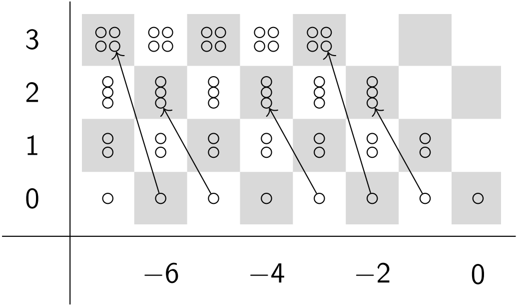
$$H^*((\mathbb{Z}/2)^2; k) \cong k[x_1, x_2] \text{ with } |x_i| = 1.$$

Note that $Q_{2^n}/H \cong D_{2^{n-1}}$.

$$H^*(D_{2^{n-1}}; k) \cong k[x_1, u, z]/(ux_1 + x_1^2 = 0)$$

with $|x_i| = |u| = 1, |z| = 2$. *Moreover,* $\mathrm{Sq}^1(z) = uz$.

$$E_2^{s,t} = H^s(Q/H; \pi_t(k^{h\mathbb{Z}/2})) \Rightarrow \pi_{t-s}(k^{hQ})$$



The Adams-graded E_2 page. $\circ = k$. Not all differentials are drawn.

Proposition

For $G = Q_8$, we have differentials

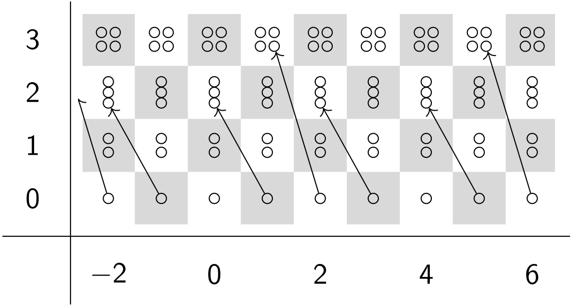
$$d_2(t) = x_1^2 + x_1x_2 + x_2^2 \qquad \text{and} \qquad d_3(t^2) = x_1^2x_2 + x_1x_2^2$$

Proposition

For $G = Q_{2^n}$, we have differentials

$$d_2(t) = u^2 + z \qquad \text{and} \qquad d_3(t^2) = uz$$

$$E_2^{s,t} = H^s(Q/H; \pi_t(k^{t\mathbb{Z}/2})) \Rightarrow \pi_{t-s}(k^{tQ})$$



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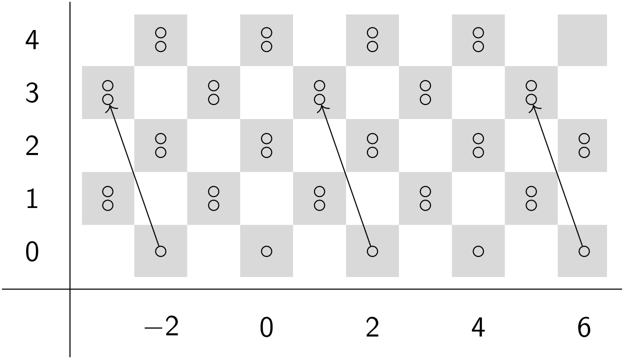
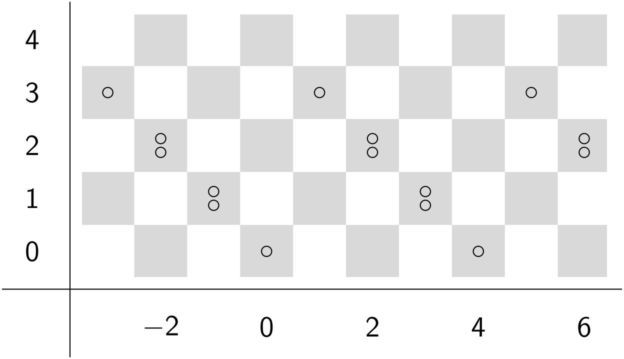


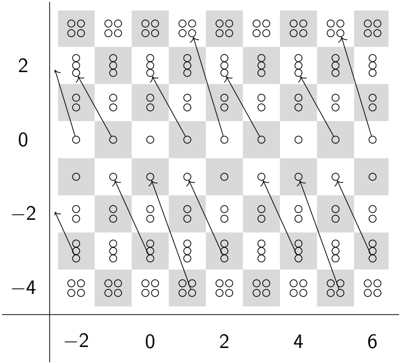
The Adams-graded E_3 page. $\circ = k$. Not all differentials are drawn

$$E_2^{s,t} = H^s(Q/H; \pi_t(k^{t\mathbb{Z}/2})) \Rightarrow \pi_{t-s}(k^{tQ})$$

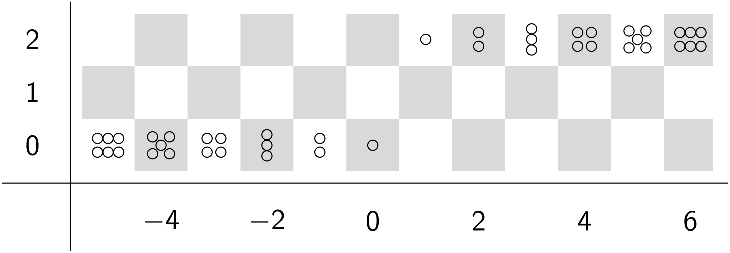


The Adams-graded $E_4 = E_\infty$ page. $\circ = k$.

$$E_2^{s,t} = \widehat{H}^s(Q/H; \pi_t(k^{t\mathbb{Z}/2})) \Rightarrow \pi_{t-s}((k^{t\mathbb{Z}/2})^{tQ/H})$$

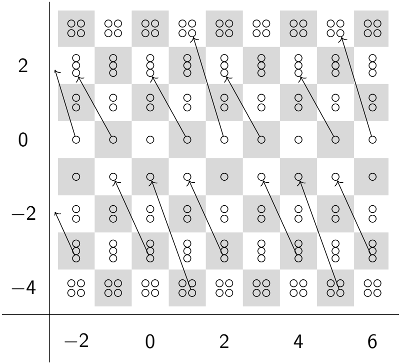


The Adams graded E_2 page of the Tate spectral sequence. $\circ = k$. Not all differentials are drawn.



The Adams graded $E_2 = E_\infty$ page of the Čech cohomology spectral sequence computing $H^*(Q/H; k)$.

$$E_2^{s,t} = \widehat{H}^s(Q/H; \pi_t(k^{t\mathbb{Z}/2})) \Rightarrow \pi_{t-s}((k^{t\mathbb{Z}/2})^{tQ/H})$$



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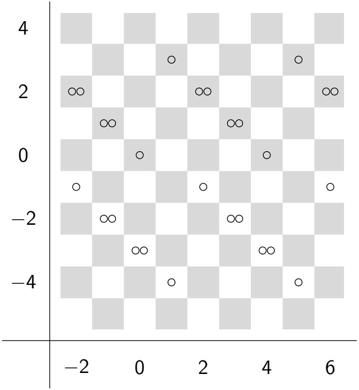
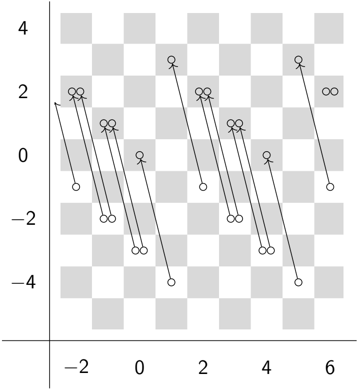


The Adams graded E_4 page of the Tate spectral sequence. $\circ = k$.

$$E_2^{s,t} = \widehat{H}^s(Q/H; \pi_t(k^{t\mathbb{Z}/2})) \Rightarrow \pi_{t-s}((k^{t\mathbb{Z}/2})^{tQ/H})$$



The Adams graded E_4 page of the Tate spectral sequence. $\circ = k$.

Corollary

The descent spectral sequence for StMod(kQ) *is the homotopy fixed point spectral sequence:*

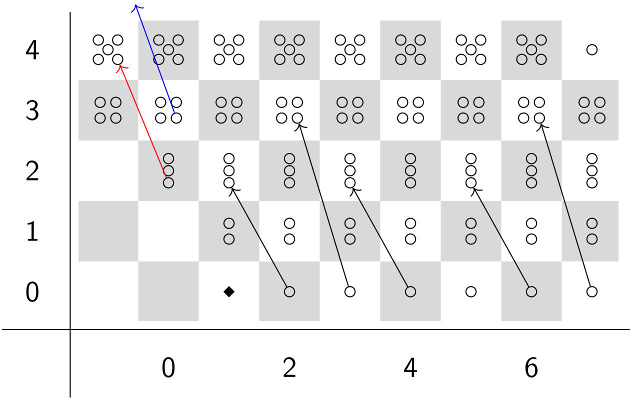
$$H^s(Q/H; \pi_t(\mathfrak{pic}(k^{t\mathbb{Z}/2}))) \Rightarrow \pi_{t-s}(\mathfrak{pic}(k^{t\mathbb{Z}/2})^{hQ/H})$$

whose abutment for $t = s$ *is* Pic(StMod(kQ)).

Proposition

The homotopy groups of $\mathfrak{pic}(k^{t\mathbb{Z}/2})$ *are given by:*

$$\pi_*(\mathfrak{pic}(k^{t\mathbb{Z}/2})) \cong \begin{cases} \mathrm{Pic}(k^{t\mathbb{Z}/2}) \cong 1 & * = 0 \\ k^\times & * = 1 \\ \pi_{*-1}(k^{t\mathbb{Z}/2}) \cong k & * \geq 2 \end{cases}$$



The Adams graded E_2 page of the HFPSS computing $\pi_*(\mathfrak{pic}(k^{t\mathbb{Z}/2})^{hQ/H})$. Not all differentials are drawn. $\circ = k$, $\blacklozenge = k^\times$.

By the construction of $\mathfrak{pic}(R)$, we have an identification of differentials $d_r^{s,t}(\mathfrak{pic}S) \cong d_r^{s,t-1}(S)$ for $t-s>0$ and $s>0$.

Theorem (Mathew-Stojanoska)

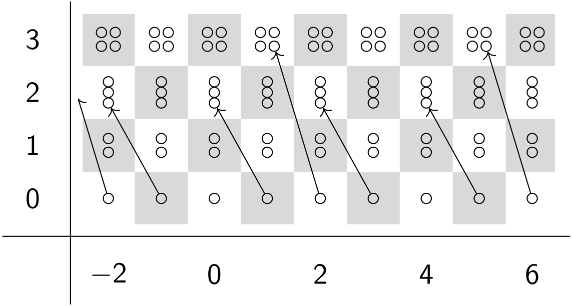
Let $R \to S$ be a G-Galois extension of E_∞ ring spectra. Then we further have an identification of differentials for $2 \leq r \leq t-1$, which yields an isomorphism

$$f : E_t^{t,t-1}(S) \xrightarrow{\cong} E_t^{t,t}(pic(S))$$

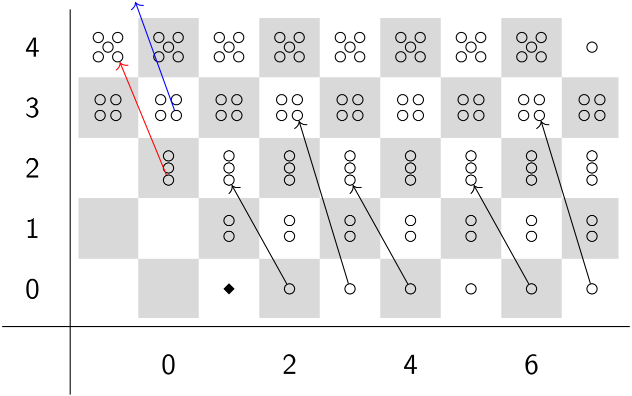
Moreover, there is a formula for the first differential outside of this range:

$$d_t^{t,t}(f(x)) = f(d_t^{t,t-1}(x) + x^2), \;\; x \in E_t^{t,t-1}(S)$$

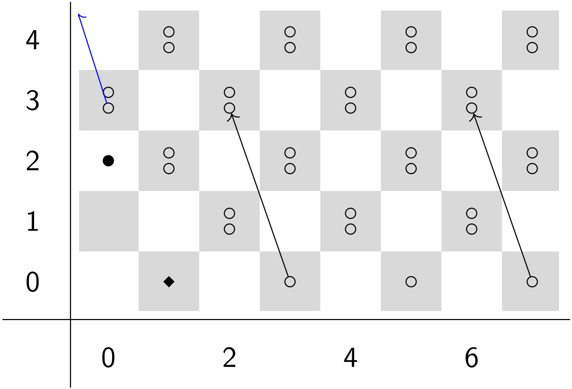
$$E_2^{s,t} = H^s(Q/H; \pi_t(k^{t\mathbb{Z}/2})) \Rightarrow \pi_{t-s}(k^{tQ})$$



The Adams-graded E_2 page. $\circ = k$. Not all differentials are drawn.



The Adams graded E_2 page of the HFPSS computing $\pi_*(\mathfrak{pic}(k^{t\mathbb{Z}/2})^{hQ/H})$. Not all differentials are drawn. $\circ = k$, $\blacklozenge = k^\times$.



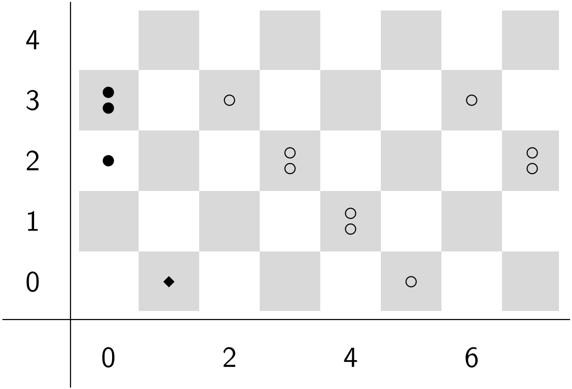
The Adams graded E_3 page of the HFPSS computing $\pi_*(\mathfrak{pic}(k^{tQ_8}))$. Not all differentials are drawn. $\circ = k$, $\blacklozenge = k^\times$, $\bullet = \mathbb{Z}/2$.

For $G = Q_8$, note that $E_3^{3,3} \cong k^2$, generated by $t^{-2}x_1x_2^2$ and $t^{-2}x_1^2x_2$. Applying the formula for the differential for $\alpha, \beta \in k$, noting that $x_1^3x_2^3 = x_1^4x_2^2 + x_1^2x_2^4$, we have

$$\begin{aligned} d_3(f(\alpha t^{-2}x_1^2x_2)) &= f(\alpha t^{-4}(x_1^4x_2^2 + x_1^3x_2^3) + f(\alpha^2 t^{-4}x_1^4x_2^2) \\ &= f(\alpha t^{-4}(x_1^4x_2^2 + (x_1^4x_2^2 + x_1^2x_2^4)) + f(\alpha^2 t^{-4}x_1^4x_2^2) \\ &= f(\alpha^2 t^{-4}x_1^4x_2^2) + f(\alpha t^{-4}x_1^2x_2^4) \end{aligned}$$

$$\begin{aligned} d_3(f(\beta t^{-2}x_1x_2^2)) &= f(\beta t^{-4}(x_1^3x_2^3 + x_1^2x_2^4) + f(\beta^2 t^{-4}x_1^2x_2^4) \\ &= f(\beta t^{-4}((x_1^4x_2^2 + x_1^2x_2^4) + x_1^2x_2^4) + f(\beta^2 t^{-4}x_1^2x_2^4) \\ &= f(\beta^2 t^{-4}x_1^2x_2^4) + f(\beta t^{-4}x_1^4x_2^2) \end{aligned}$$

For an element to be in the kernel, we then must simultaneously have the expressions $\alpha + \beta^2 = 0$ and $\beta + \alpha^2 = 0$.



The Adams graded E_4 page of the HFPSS computing $\mathfrak{pic}((k)^{tQ_8})$, where k has a cube root of unity. $\circ = k$, $\bullet = \mathbb{Z}/2$, $\blacklozenge = k^\times$.

For $G = Q_{2^n}$, note that $E_3^{3,3} \cong k^2$, generated by $t^{-2}uz$ and $t^{-2}x_1z$. Applying the formula for the differential for $\alpha, \beta \in k$, noting that $ux_1 = x_1^2$ in the E_3 page, we have

$$\begin{aligned} d_3(f(\alpha t^{-2}uz)) &= f(\alpha u^2 z^2 t^{-4}) + f(\alpha^2 u^2 z^2 t^{-4}) \\ &= f((\alpha + \alpha^2)u^2 z^2 t^{-4}) \end{aligned}$$

$$\begin{aligned} d_3(f(\beta t^{-2}x_1 z)) &= f(\beta(ux_1 z^2)t^{-4}) + f(\beta^2 x_1 z^2 t^{-4}) \\ &= f((\beta + \beta^2)x_1 z^2 t^{-4}) \end{aligned}$$

For an element to be in the kernel, we must simultaneously have the expressions $\alpha + \alpha^2 = 0$ and $\beta + \beta^2 = 0$.

Theorem (van de Meer-W.)

Let ω denote a cube root of unity.

$$\mathrm{Pic}(\mathrm{StMod}(kQ_8)) \cong \begin{cases} \mathbb{Z}/4 & \text{if } \omega \notin k \\ \mathbb{Z}/4 \oplus \mathbb{Z}/2 & \text{if } \omega \in k \end{cases}$$

Theorem (van de Meer-W.)

Let $n \geq 4$.

$$\mathrm{Pic}(\mathrm{StMod}(kQ_{2^n})) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$$

Future Directions

- **Generalizations** - Compute Pic(StMod(kG)) for G dihedral and semi-dihedral, or for extraspecial and almost-extraspecial p-groups.
- **Tensor-Triangulated Geometry** - Compute Pic(Γ_p(StMod(kG))), where Γ_p(StMod(kG)) denotes a thick or localizing tensor-ideal subcategory of StMod(kG).
- Categorify the Dade group of endo-permutation modules.
- Further HFPSS or Tate spectral sequence calculations.

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