Picard Groups of Stable Module Categories

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GROOT Summer Seminar 2020

Slides can be found at
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Let k be a field of positive characteristic p, and let G be a finite group such that $p \mid |G|$.

We are interested in studying $\mathrm{Mod}(kG)$, the category of modules over the group ring kG. This is the setting of **modular representation theory**.

In this setting, Maschke's theorem does not apply:

Theorem (Maschke)

The group algebra kG is semisimple iff the characteristic of k does not divide the order of G.

In particular, one corollary is that not every module in $\mathrm{Mod}(kG)$ is projective.

Definition

The **stable module category** $\mathrm{StMod}(kG)$ has objects kG-modules, and has morphisms

$$\underline{\mathrm{Hom}}_{kG}(M, N) = \mathrm{Hom}_{kG}(M, N)/\mathrm{PHom}_{kG}(M, N)$$

where $\mathrm{PHom}_{kG}(M, N)$ is the linear subspace of maps that factor through a projective module.

Definition

We say two maps $f, g : M \to N$ are **stably equivalent** if $f - g$ factors through a projective module.

Proposition

StMod(kG) *is the homotopy category of a stable model category structure on* Mod(kG).

The weak equivalences are the stable equivalences.

The fibrations are surjections. The acyclic fibrations are surjections with projective kernel.

The suspension of a module M is denoted $\Omega^{-1}(M)$, and is constructed as the cofiber of an inclusion into an injective module:

$$M \hookrightarrow I \to \Omega^{-1}(M)$$

Proposition

StMod(kG) *is a stable symmetric monoidal ∞-category.*

From now on, we restrict our attention to the case that G is a finite p-group, so that the following theorem holds:

Theorem (Mathew)

There is an equivalence of symmetric monoidal ∞-categories

$$\mathrm{StMod}(kG) \simeq \mathrm{Mod}(k^{tG})$$

Remark

The proof goes through the identifications

$$\mathrm{Ind}(\mathrm{Fun}(BG, \mathrm{Perf}(k))) \cong \mathrm{Mod}(k^{hG})$$

and for $A = F(G_+, k)$,

$$\mathrm{StMod}(kG) \cong L_{A^{-1}}\mathrm{Ind}(\mathrm{Fun}(BG, \mathrm{Perf}(k)))$$

The spectrum $k^{hG} \simeq F(BG_+, k)$ is the E_∞ ring of cochains on BG with coefficients in k. It is also the G-homotopy fixed points of k with the trivial action.

Proposition

There is an isomorphism of graded rings

$$\pi_*(k^{hG}) \cong H^{-*}(G; k)$$

Example

For $p = 2$, $\pi_*(k^{h(\mathbb{Z}/2)^n}) \cong k[x_1, \ldots, x_n]$ with $|x_i| = 1$.

For p odd, $\pi_*(k^{h(\mathbb{Z}/p)^n}) \cong k[x_1, \ldots, x_n] \otimes \Lambda(y_1, \ldots, y_n)$ with $|x_i| = 2, |y_i| = 1$.

Theorem

We have the homotopy fixed point spectral sequence, which takes in input the spectrum R with a G-action, and computes $\pi_*(R^{hG})$:

$$E_2^{s,t}(R) = H^s(G; \pi_t(R)) \Rightarrow \pi_{t-s}(R^{hG})$$

There is also a notion of **homotopy orbits** k_{hG}, and homotopy orbit spectral sequence.

Proposition

There is an isomorphism

$$\pi_*(k_{hG}) \cong H_*(G; k)$$

Just like there is a norm map in group cohomology

$$N_G : H_*(G;k) \to H^*(G;k)$$

there is a norm map $N_G : k_{hG} \to k^{hG}$.

And just as one can stitch together group homology and cohomology via the norm map to form Tate cohomology,

$$\widehat{H}^i(G;k) \cong \begin{cases} H^i(G;k) & i \geq 1 \\ \operatorname{coker}(N_G) & i = 0 \\ \ker(N_G) & i = -1 \\ H_{-i-1}(G;k) & i \leq -2 \end{cases}$$

Definition

The **Tate fixed points** are the cofiber of the norm map:

$$k_{hG} \xrightarrow{N_G} k^{hG} \to k^{tG}$$

We have the Tate fixed point spectral sequence, which takes in input the spectrum R with a G-action, and computes $\pi_*(R^{tG})$:

$$E_2^{s,t}(R) = \widehat{H}^s(G; \pi_t(R)) \Rightarrow \pi_{t-s}(R^{tG})$$

Remark

The multiplication of elements in negative degrees in $\pi_*(k^{tG})$ is the same as the multiplication in $\pi_*(k^{hG})$.

Multiplication by elements in positive degrees is more complicated. For example, if G is an elementary abelian group of p-rank ≥ 2,

$$\pi_n(k^{tG}) \cdot \pi_m(k^{tG}) = 0$$

for all $n, m > 0$.

Theorem (Mathew)

For G a finite p-group, there is an equivalence of symmetric monoidal ∞-categories

$$\mathrm{StMod}(kG) \simeq \mathrm{Mod}(k^{tG})$$

Remark

Historically, the study of $\mathrm{StMod}(kG)$ *was very closely related to the study of group and Tate cohomology.*

Ernie break

Definition

The **Picard group** of a symmetric monoidal (∞-)category $(\mathcal{C}, \otimes, 1)$, denoted $\mathrm{Pic}(\mathcal{C})$, is the set of isomorphism classes of invertible objects X, with

$$[X] \cdot [Y] = [X \otimes Y]$$

$$[X]^{-1} = [\mathrm{Hom}_{\mathcal{C}}(X, 1)]$$

Example

The following are examples of stable symmetric monoidal ∞-categories:

(a) $(\mathrm{Sp}, \wedge, \mathbb{S}, \Sigma)$

(b) $(D(R), \hat{\otimes}_R, R[0], -[1])$ for R a commutative ring.

(c) $(\mathrm{Mod}(R), \wedge_R, R, \Sigma)$ for R a commutative ring spectrum.

(d) $(\mathrm{StMod}(kG), \otimes_k, k, \Omega^{-1})$ in modular characteristic.

Theorem (Hopkins-Mahowald-Sadofsky)

$\mathrm{Pic}(\mathrm{Sp}) \cong \mathbb{Z}$

That is, for any $X \in \mathrm{Pic}(\mathrm{Sp})$, *we have that* $X \cong \Sigma^i \mathbb{S}$ *for some* $i \in \mathbb{Z}$.

Theorem (Dade)

Let E *denote an abelian* p*-group. Then* $\mathrm{Pic}(\mathrm{StMod}(kE))$ *is cyclic.*

Given a symmetric monoidal ∞-category $\mathcal{C}$, one can do better than the Picard group:

Definition

The **Picard space** $\mathcal{P}\mathrm{ic}(\mathcal{C})$ is the ∞-groupoid of invertible objects in $\mathcal{C}$ and isomorphisms between them.

This is a group-like E_∞-space, and so we equivalently obtain the connective **Picard spectrum** $\mathfrak{pic}(\mathcal{C})$.

Proposition

The functor $\mathfrak{pic} : \mathrm{Cat}^{\otimes} \to \mathrm{Sp}_{\geq 0}$ *commutes with limits and filtered colimits.*

Example

Let R be an E_∞-ring spectrum. The homotopy groups of $\mathfrak{pic}(R)$ are given by:

$$\pi_*(\mathfrak{pic}(R)) \cong \begin{cases} \mathrm{Pic}(R) & * = 0 \\ (\pi_0(R))^\times & * = 1 \\ \pi_{*-1}(\mathfrak{gl}_1(R)) \cong \pi_{*-1}(R) & * \geq 2 \end{cases}$$

Note that the isomorphism $\pi_*(\mathfrak{gl}_1(R)) \cong \pi_*(R)$ for $* \geq 1$ is usually not compatible with the ring structure.

Ernie break

Ernie's 2019 Halloween Costume

Theorem (Mathew-Stojanoska)

If $f : R \to S$ is a faithful G-Galois extension of E_∞ ring spectra, then we have an equivalence of ∞-categories

$$\mathrm{Mod}(R) \cong \mathrm{Mod}(S)^{hG}$$

Corollary

We have the homotopy fixed point spectral sequence, which takes in input the spectrum $pic(S)$ and has E_2 page:

$$H^s(G; \pi_t(pic(S)) \Rightarrow \pi_{t-s}(pic(S)^{hG})$$

whose abutment for $t = s$ is $\mathrm{Pic}(R)$.

Definition

A map $f : R \to S$ of E_∞-ring spectra is a G-**Galois extension** if the maps

(i) $i : R \to S^{hG}$

(ii) $h : S \otimes_R S \to F(G_+, S)$

are weak equivalences.

Definition

A G-Galois extension of E_∞-ring spectra $f : R \to S$ is said to be **faithful** if the following property holds:

If M is an R-module such that $S \otimes_R M$ is contractible, then M is contractible.

Example

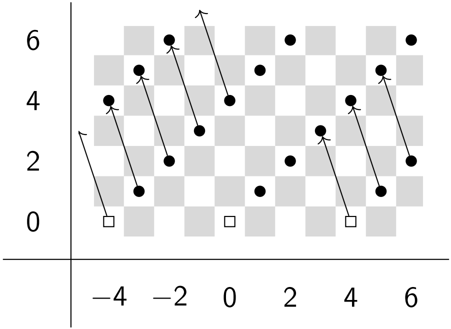
$KO \to KU$ is a faithful $\mathbb{Z}/2$-Galois extension of ring spectra.

Note that $\pi_*(KU) \cong \mathbb{Z}[u^{\pm 1}]$ with $|u| = 2$., which is very homologically simple. On the other hand, $\pi_*(KO)$ is more complicated.

Proposition (Rognes)

A G-Galois extension of E_∞-ring spectra $f : R \to S$ is faithful if and only if the Tate construction S^{tG} is contractible.

$$E_2^{s,t} = H^s(\mathbb{Z}/2; \pi_t(KU)) \Rightarrow \pi_{t-s}(KU^{h\mathbb{Z}/2})$$



The Adams graded $\mathbb{Z}/2$-HFPSS computing $\pi_*(KU^{h\mathbb{Z}/2}) \cong \pi_*(KO)$.
$\square = \mathbb{Z}$, $\bullet = \mathbb{Z}/2$.

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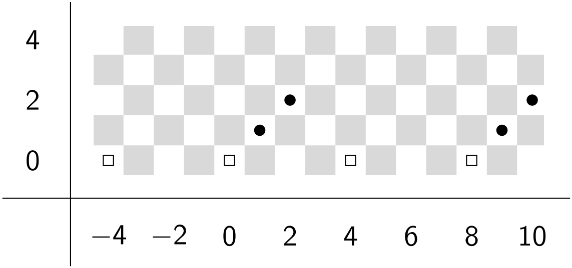

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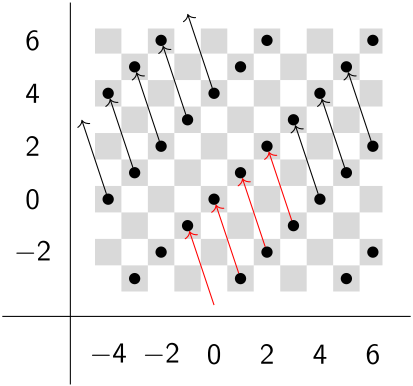
$$E_2^{s,t} = \widehat{H}^s(\mathbb{Z}/2; \pi_t(KU)) \Rightarrow \pi_{t-s}(KU^{t\mathbb{Z}/2})$$



The Adams graded $\mathbb{Z}/2$-Tate SS computing $\pi_*(KU^{t\mathbb{Z}/2})$. $\bullet = \mathbb{Z}/2$.

Let $R \to S$ be a G-Galois extension of E_∞-rings.

Corollary

We have the homotopy fixed point spectral sequence, which takes in input the spectrum $pic(S)$ *and has* E_2 *page:*

$$H^s(G; \pi_t(pic(S))) \Rightarrow \pi_{t-s}(pic(S)^{hG})$$

whose abutment for $t = s$ *is* $\mathrm{Pic}(R)$*.*

Theorem (Mathew-Stojanoska)

If $t - s > 0$ *and* $s > 0$ *we have an equality of HFPSS differentials*

$$d_r^{s,t}(\mathfrak{pic} S) \cong d_r^{s,t-1}(S)$$

Furthermore, this equality also holds whenever $2 \leq r \leq t - 1$*.*

Example

We will calculate $\mathrm{Pic}(KO)$ using the fact that $KO \to KU$ is a faithful $\mathbb{Z}/2$-Galois extension of ring spectra.

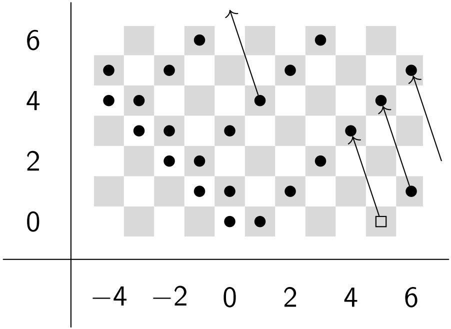
Recall that $\pi_*(KU) \cong \mathbb{Z}[u^{\pm 1}]$, with $|u| = 2$. Since KU is even periodic with a regular noetherian π_0,

$$\mathrm{Pic}(KU) \cong \mathrm{Pic}(\pi_*(KU)) \cong \mathbb{Z}/2$$

The homotopy groups of $\mathfrak{pic}(KU)$ are given by:

$$\pi_*(\mathfrak{pic}(R)) \cong \begin{cases} \mathrm{Pic}(KU) \cong \mathbb{Z}/2 & * = 0 \\ (\pi_0(KU))^\times \cong \mathbb{Z}/2 & * = 1 \\ \pi_{*-1}(KU) & * \geq 2 \end{cases}$$

$$E_2^{s,t} = H^s(\mathbb{Z}/2; \pi_t(\mathfrak{pic}(KU))) \Rightarrow \pi_{t-s}((\mathfrak{pic}(KU))^{h\mathbb{Z}/2})$$



The Adams graded $\mathbb{Z}/2$-HFPSS computing $\pi_*((\mathfrak{pic}(KU))^{h\mathbb{Z}/2})$. $\square = \mathbb{Z}$, $\bullet = \mathbb{Z}/2$. Not all differentials are drawn.

Example

Let G be a finite p-group and H a normal subgroup. Then

$$k^{hG} \to k^{hH} \quad \text{and} \quad k^{tG} \to k^{tH}$$

are G/H-Galois extensions of ring spectra. Note however that these Galois extension are not necessarily faithful.

Remark (Work in progress)

For Q a quaternion group, and $\mathbb{Z}/2 = Z(Q)$,

$$k^{tQ} \to k^{t\mathbb{Z}/2}$$

is almost faithful.

Remark

This comes from taking cochains $F((-)_+, k)$ of the fiber sequence

$$G/H \to BH \to BG$$

However, to see that $S \otimes_R S \simeq F((G/H)_+, S)$, one needs the convergence of the mod p Eilenberg-Moore spectral sequence.

Theorem (Mathew)

Let E be an elementary abelian p-group of rank n. Then we have a short exact sequence $\mathbb{Z}^n \to \mathbb{Z}^n \to E$. This yields a fiber sequence

$$B\mathbb{Z}^n \to BE \to B^2\mathbb{Z}^n$$

Taking cochains, we have faithful $\mathbb{T}^n$-Galois extensions

$$k^{h\mathbb{T}^n} \to k^{hE} \quad \text{and} \quad k^{t\mathbb{T}^n} \to k^{tE}$$

Remark

In this case, we understand $\pi_*(k^{h\mathbb{T}^n}) \cong k[x_1, \cdots, x_n]$ well. So we need to do ***reverse*** *Galois descent.*

That is, for $R \to S$ is a faithful $\mathbb{T}^n$-Galois extension, when does $M \in \mathrm{Pic}(S)$ descend from $M \in \mathrm{Pic}(R)$?

Theorem (Mathew)

Let E be an elementary abelian p-group of rank n. Then we have a short exact sequence $\mathbb{Z}^n \to \mathbb{Z}^n \to E$. This yields a fiber sequence

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Remark

In this case, we understand $\pi_*(k^{h\mathbb{T}^n}) \cong k[x_1, \cdots, x_n]$ well. So we need to do ***reverse*** *Galois descent.*

That is, for $R \to S$ is a faithful $\mathbb{T}^n$-Galois extension, when does $M \in \mathrm{Pic}(S)$ descend from $M \in \mathrm{Pic}(R)$?

Theorem (Dade, Mathew)

Let E denote an abelian p-group. Then $\mathrm{Pic}(\mathrm{StMod}(kE))$ *is cyclic.*

Proof.

- Show $\mathrm{Pic}(k^{t\mathbb{T}^n}) \cong C$ is cyclic.
- Show that for $R \to S$ a faithful $\mathbb{T}^n$-Galois extension, $M \in \mathrm{Pic}(S)$ descends from $M \in \mathrm{Pic}(R)$ iff for every $a \in \pi_1(\mathbb{T}^n)$, the induced monodromy automorphism $a : M \to M$ is the identity.
- Show that for $k^{t\mathbb{T}^n} \to k^{tE}$, the monodromy is always trivial.
- Hence we have a surjection $C \to \mathrm{Pic}(\mathrm{StMod}(kE))$.

□