NUMBER GAMES ANSWER KEY, SMMG SPRING 2020

RICHARD WONG

Abstract. We will explore and play with some of the weird and interesting facts and formulas surrounding these cool types of numbers called pandigital numbers and Friedman numbers. In particular, we will learn about a surprisingly good approximation of a number that a lot of people call e.

1. Pandigital Numbers

Definition 1.1. A **pandigital number** is a number that uses each digit 0-9 **exactly once** in the significant digits of its decimal representation.

Definition 1.2. A **pandigital number with redundant digits** is a number that uses each digit 0-9 **at least once** in the significant digits of its decimal representation.

1. What are the first 5 smallest pandigital numbers?
   - 1023456789, 1023456798, 1023456879, 1023456897, 1023456978
2. What is the largest pandigital number?
   - 9876543210
3. Can you find a prime pandigital number?
   - No. Every pandigital number is divisible by 3 and 9.
4. **Extremely Hard:** Can you find a prime pandigital number with redundant digits?
   - 10123457689.
5. How many pandigital numbers are there?
   - There are $9 \times 9! = 3265920$ pandigital numbers. Remember that 0 cannot be a leading digit.

The concept of pandigital numbers can be extended beyond decimal representations, which is also known as **base 10**. You might have heard of base 2, or binary.

7. How would you define a pandigital number in a different base, such as binary?
   - A **pandigital number in base** b is a number that uses each digit *in base* b exactly once in the significant digits in *base* b.
8. How many digits are in binary? How many digits are in base b?
   - There are 2 digits in binary, and b digits in base b.
9. What are the first 5 smallest pandigital numbers (with redundant digits) in binary?
   - There's only one pandigital number in binary - 10. With redundant digits, we then have 100, 101, 110, 1000.
10. How many pandigital numbers are there in binary?
    - Only 1. (the binary number 10).
11. How many pandigital numbers are there in base b?
    - There are $(b-1) \times (b-1)!$ pandigital numbers. There are $(b-1)!$ ways to arrange the non-zero digits, and $b-1$ places to place the 0 digit.

2. Friedman Numbers

Definition 2.1. A **Friedman number** is a number that can be non-trivially expressed as a formula using each of its significant digits exactly once, along with the operations $(+, -, \times, \div)$, additive inverses, parentheses, and exponentiation.

1. What is the smallest Friedman number?
   - $25 = 5^2$.
2. Show that 121, 125, 126, 127, and 128 are Friedman numbers.
   - $121 = 11^2$.
   - $125 = 5^{(1+2)}$.
   - $126 = 6 \times 21$.
   - $127 = 2^7 - 1$.
   - $128 = 2^{(8-1)}$
3. **Hard:** Prove that there is only one two-digit Friedman number.
   - Note that we can write numbers x as $x = 10m + n$.
   - Note that it is always true that $m + n < x$, since $n < 10$.
   - Similarly, note that it is always true that $m \times n < x$,.
   - Repeat for $m - n$, $n - m$, $m \div n$, and $n \div m$.
   - Therefore, we only need to consider $x = m^n$ and $x = n^m$.
4. Show that 1395 is a Friedman number. There are two different ways this is true!
   - $1395 = 5 \times 9 \times 31 = 15 \times 93$.
5. **BONUS:** What can you say about Friedman numbers in base b?
   - You can reiterate the argument of **3.** to determine all of the 2-digit base b Friedman numbers, replacing 10 with b.

To pass the time, you might want to play the **24 game**. To play the game, you are given four numbers. The goal is to write the number 24 as a formula using each of those numbers exactly once, along with the operations $(+, -, \times, \div)$, and parentheses. You can play this with a deck of cards, but then you're not always guaranteed a solution!

A. 3, 4, 5, 5.
   - $(5 \times 5) - (4 - 3)$.

B. 2, 2, 5, 7.
   - $(2 \times 5) + (2 \times 7)$.

C. 2, 4, 5, 11.
   - $(2 \times 4) + 5 + 11$

D. 5, 5, 1, 12.
   - $((5 \div 5) + 1) \times 12$.

E. 1, 3, 4, 6. **Hard Challenge**: Show there's only one possible solution!
   - $6 \div (1 - (3 \div 4))$.

3. Approximating e.

The number e is a mathematical constant that is used a lot of mathematics, including calculus, probability theory, and even calculating compound interest. It is an irrational, and even transcendental number, and is approximately

$$e = 2.718281828459045\ldots$$

Here are some formulas to approximate e: Try them out, and **see how long it takes for them to get to within 5 significant digits of e.**

1. As a sum:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

   - We reach it with $k = 7$.

2. As a limit:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

   - We reach it with approximately $n = 20000$. Graph the function $\left(1 + \frac{1}{x}\right)^x$! Can you show that it is equivalent to the previous one (using binomial expansion and analysis methods)?

3. As a continued fraction:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \cdots}}}}}$$

   - We reach it with the 7th level. Compare this to the first formulation using Euler's continued fraction formula!

4. It is conjectured [RPM+19] that the following continued fraction is equal to e, but this has not yet been proven!

$$3 + \cfrac{-1}{4 + \cfrac{-2}{5 + \cfrac{-3}{6 + \cfrac{-4}{7 + \cfrac{-5}{8 + \cdots}}}}}$$

   - We also reach it with the 7th level.

References

[Cop14] Ed Copeland. Friedman Numbers - Numberphile, 2014.

[Gri13] James Grime. Why 381,654,729 is awesome - Numberphile, 2013.

[Gri16] James Grime. Incredible Formula - Numberphile, 2016.

[RPM+19] Gal Raayoni, George Pisha, Yahel Manor, Uri Mendlovic, Doron Haviv, Yaron Hadad, and Ido Kaminer. The ramanujan machine: Automatically generated conjectures on fundamental constants, 2019.

Department of Mathematics, University of Texas at Austin, Austin, TX 78751

Current address: Department of Mathematics, University of Texas at Austin, Austin, TX 78751

E-mail address: `richard.wong@math.utexas.edu`