The nilpotency of elements of the stable homotopy groups of spheres

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The stable homotopy groups of spheres $\pi_*(\mathbb{S})$ has a ring structure, given by either composition or smash product of spectra ($\mathbb{S}$ is a ring spectrum). These are equivalent by an Eckmann-Hilton argument.

Theorem (Nishida)

Any element in the positive stem of the stable homotopy groups of spheres is nilpotent.
That is, given $\alpha \in \pi_k(\mathbb{S})$ with $k > 0$, there exists an integer n such that $\alpha^n = 0$.

- In this paper, Nishida gives two different proofs. The first proof yields nilpotence for elements $\alpha \in \pi_k(\mathbb{S})$ of order p.
- The ideas in this proof were built on by Devinatz-Hopkins-Smith, and generalized to the Nilpotence theorem:

Theorem (Nilpotence theorem)

Let R be a ring spectrum and let

$$\pi_*(R) \xrightarrow{h} MU_*(R)$$

be the Hurewicz map. If $h(\alpha) = 0$, then $\alpha \in \pi_*(R)$ is nilpotent.

- However, his second proof, for any element $\alpha \in \pi_k(\mathbb{S})$, relies heavily on the **Araki-Kudo-Dyer-Lashof operations**, which were later encoded into the notion of an $\mathbb{H}_\infty$**-ring spectrum**.
- Moreover, the ideas in this proof lead to May's Nilpotence Conjecture (which was recently proven by Mathew-Naumann-Noel):

Theorem (May's Nilpotence Conjecture)

Suppose that R is an $\mathbb{H}_\infty$-ring spectrum and $x \in \pi_*(R)$ satisfies $p^m x = 0$ for some integer m.
Then x is nilpotent if and only if its Hurewicz image in $(H\mathbb{F}_p)_*(R)$ is nilpotent.

- While anachronistic, it is not too hard to prove Nishida's result using the nilpotence machinery of Devinatz-Hopkins-Smith. Note however, this proof is non-constructive.
- The hard part is getting good estimates on the bounds of nilpotent elements.
- In Nishida's first proof, for an element $\alpha \in \pi_k(\mathbb{S})$ of order p, the bound on the exponent is roughly $(k+1)\frac{p}{2p-3}$.
- However, in his second proof, for an element $\alpha \in \pi_k(\mathbb{S})$ of order p, we obtain a much worse estimate of roughly $2^{\lfloor \frac{k+1}{2} \rfloor}$ for $p = 2$, and $p^{\lfloor \frac{k+1}{p-1} \rfloor + 1}$ for odd p.

Theorem (Nilpotence theorem, ring spectrum form)

Let R be a ring spectrum and let

$$\pi_*(R) \xrightarrow{h} MU_*(R)$$

be the Hurewicz map (induced by $\mathbb{S} \to MU$). If $h(\alpha) = 0$, then $\alpha \in \pi_*(R)$ is nilpotent.

- This is to say, the kernel of the map h consists of nilpotent elements.
- Another way to think about this result is that MU detects nilpotence - that is, if a map $f : X \to Y$ from a finite spectrum X is trivial in MU homology, then f is nilpotent.

Corollary (Nishida's Theorem)

For $k > 0$, every element of $\pi_k(\mathbb{S})$ is nilpotent.

Proof.

- Positive degree elements in $\pi_*(\mathbb{S})$ are torsion. [Serre] So $x \in \pi_k(\mathbb{S})$ for $k > 0$ is torsion, and hence the image of x in $\pi_*(MU)$ is also torsion.
- But $\pi_*(MU) \cong \mathbb{Z}[x_1, x_2, \dots]$ with $|x_i| = 2i$ is torsion free [Milnor-Quillen], so the image of x is zero. By the Nilpotence theorem, this implies that x is nilpotent.

$\square$

The key idea is the *extended n-th power construction*.

Construction

Given a CW complex X, note that a subgroup G of S_n acts on $X^{\wedge n}$ by permuting the factors. Then one forms the extended n-th power functor as follows:

$$D_G(X) := (X^{\wedge n})_{hG} = EG_+ \wedge_G X^{\wedge n}$$

This has a skeletal filtration induced by the skeletal filtration of EG_+.

We will consider the cases $G = S_n$ or G a p-Sylow subgroup of S_n, and write the construction $D_n(X)$ or $D_p(X)$ respectively.

Remark

Note that on the 0-skeleton, $D_G^{(0)}(X) = X^{\wedge n}$*, and* $D_G^{(0)}(f) = f^{\wedge n}$

- Therefore, the key idea in proving the theorem is understanding the stable homotopy type of $D_G(X)$ for $X = S^k$.
- Since we are looking at elements of order p, it is also useful to study the n-th power construction for $X = S^k \bigcup_p e^{k+1}$, the Moore space of type $(\mathbb{Z}/p, k)$.

Proof sketch (Ravenel)

Suppose $\alpha \in \pi_{2k}(\mathbb{S})$ with $k > 0$. Since it is of order p, that means we have an extension

$$\begin{array}{ccc} \Sigma^{2k}\mathbb{S} & \xrightarrow{\alpha} & \mathbb{S} \\ \downarrow & \nearrow & \\ \Sigma^{2k}(D_1) & & \end{array}$$

Where D_1 is the mod p Moore spectrum (a finite spectrum built as $D_1 = \mathbb{S} \bigcup_p e^1$). The construction generalizes to an extension

$$\begin{array}{ccc} \Sigma^{2kn}\mathbb{S} & \xrightarrow{\alpha^n} & \mathbb{S} \\ \downarrow & \nearrow & \\ \Sigma^{2kn}(D_n) & & \end{array}$$

- What Nishida was able to show is that the map $D_n \to H\mathbb{Z}/p$ is an equivalence through a range of dimensions that increases with n.
- Therefore, we can choose a minimal n such that this range of dimensions contains $2k$. Then consider the commutative diagram

$$\begin{array}{ccccc} \Sigma^{2k(n+1)} & \xrightarrow{\alpha} & \Sigma^{2k}\mathbb{S} & \xrightarrow{\alpha^n} & \mathbb{S} \\ & \searrow^{q} & \downarrow & \nearrow & \\ & & \Sigma^{2k}(D_n) & & \end{array}$$

- Since $\Sigma^{2k}(D_n) \cong \Sigma^{2k} H\mathbb{Z}/p$ in this range, we know that the map q is nullhomotopic (there is no homotopy in the $2k(n+1)$ dimension).
- Therefore it follows that the composition is nullhomotopic, and hence α^{n+1} is nullhomotopic as desired.

Construction (Definition 1.3)

Suppose we have a map $i : S^k \to X$ representing a generator in homology. Then we can space-wise form a spectrum D_X by

$$(D_X)_{n(k+1)+i} = D_n(X) \wedge S^i \text{ for } 0 \leq i < k+1$$

with the usual suspension structure maps for $0 \leq i < k$, and with structure map $g_n : D_n(X) \wedge S^k \xrightarrow{1 \wedge i} D_n(X) \wedge X \xrightarrow{\mu_{n,1}} D_{n+1}(X)$

Our goal is to show that for $X = M_k := S^k \bigcup_p e^{k+1}$, the mod p Moore space, the spectrum D_{M_k} has the same mod p homotopy type as a wedge of Eilenberg-Maclane spectra.

- We are interested in computing $H_*(D_n(X), \mathbb{Z}/p)$.
- In particular, we will observe that there is a monomorphism $H_i(D_{n-1}(X) \wedge S^k) \to H_i(D_n(X))$, and an isomorphism in a range varying with n.
- This will gives us a stable range in which D_{M_k} is equivalent to a wedge of $H\mathbb{Z}/p$.

Theorem (Barratt-Eccles)

If X is connected, then there exists a natural splitting

$$\tilde{H}_*(\Gamma^+(X); \mathbb{Z}/p) \cong \oplus_n \tilde{H}_*(D_n(X); \mathbb{Z}/p)$$

Recall that $\Gamma^+(X)$ is the free monoid functor from topological spaces to simplicial monoids.

$$\Gamma^+(X) = (\bigcup ES_n \times X^n)/\sim$$

Where $(g, x_1, \ldots x_n) \sim (g, x_{\sigma(1)}, \cdots, x_{\sigma(n)})$ for $\sigma \in S_n$ and $(g, x_1, \ldots, x_{n-1}, *) \sim (Tg, x_1, \ldots, x_{n-1})$ for $T : ES_n \to ES_{n-1}$ an S_{n-1}-equivariant map.

Proposition

Note that $\pi_0(\Gamma^+(X)) \cong \mathbb{Z}^+(\pi_0(X))$. That is, the monoid of components is the free abelian monoid on the pointed set $\pi_0(X)$.

- The reason that Barratt-Eccles consider this functor is because they would like to construct a model for the infinite loop space $QX = \Omega^\infty \Sigma^\infty X$.
- However, the previous proposition implies that in general $\Gamma^+(X)$ fails to be a model for QX.
- Nevertheless, they define a free group functor ΓX to be the universal (simplicial) group of the (simplicial) monoid $\Gamma^+(X)$. This functor is universal with respect to homomorphisms from monoids to groups. That is, given a monoid homomorphism $M \to G$, there is a unique group homomorphism $UM \to G$.

Theorem (Barrat-Eccles-Quillen)

$\Gamma(X) \simeq Q(X)$. *Furthermore, if X is connected, then* $\Gamma^+(X) \simeq \Gamma(X) \simeq Q(X)$.

Theorem (Barrat-Eccles)

If X is connected, then there exists a natural splitting

$$\tilde{H}_*(Q(X);\mathbb{Z}/p) \cong \tilde{H}_*(\Gamma^+(X);\mathbb{Z}/p) \cong \oplus_n \tilde{H}_*(D_n(X);\mathbb{Z}/p)$$

Proof.

Observe we have a filtration $\Gamma_n(X) = (\bigcup^n ES_i \times X^i)/\sim$. Furthermore, by construction we have a cofibration sequence

$$\Gamma_{n-1}(X) \to \Gamma_n(X) \to D_n(X)$$

which gives us the desired splitting. □

The upshot is that we understand $\tilde{H}_*(Q(X);\mathbb{Z}/p)$ thanks to the work of Dyer-Lashof-Araki-Kudo.

For $p = 2$, [p odd], given an infinite loop space B, there exist natural stable homomorphisms $Q^i : H_*(B, \mathbb{Z}/p) \to H_*(B, \mathbb{Z}/p)$ of degree i [$2i(p-1)$] such that

1. $Q^0(1) = 1$, $Q^i(1) = 0$ for $i > 1$, where $1 \in H_0(B, \mathbb{Z}/p)$ is the unit element.
2. $Q^i(x) = 0$ if $i < deg(x)$ [$2i < deg(x)$].
3. $Q^i(X) = x^p$ if $i = deg(x)$ [$2i = deg(x)$].
4. Q^i satisfy the Cartan formula, Adem relations, and also the Nishida relations.

One should think of these as extended power operations.

Recall that an admissible sequence is a sequence $I = (s_1, \ldots, s_k)$ with $2s_j \geq s_{j+1}$. The degree is defined $d(I) = \sum s_j$, and the excess is defined as $e(I) = s_1 - \sum s_j$.

Theorem (Theorem 2.3, Dyer-Lashof)

If X is connected, then $H_*(Q(X), \mathbb{Z}/p)$ is isomorphic to a free commutative graded algebra generated by all $Q^I x_j$, where x_j is a basis of $\tilde{H}_*(X, \mathbb{Z}/p)$, and I is an admissible sequence with $e(I) > deg(x_j)$.

We will use this result to describe $H_*(D_n(X); \mathbb{Z}/p)$:

- Let $x = \prod Q_i^I(x_i)$ be a monomial. We define the height of x to be $h(x) = \sum p^{l(I_i)}$, and define $h(1) = 0$.
- Let $A_n(X)$ be the submodule of $H_*(Q(X))$ spanned by all monomials of height n.

Proposition (Proposition 2.4)

If X is connected, then $H_*(D_n(X); \mathbb{Z}/p) \cong A_n(X)$.

Proof.

The idea is to show that $H_*(\Gamma_n(X))$ is spanned by the monomials of height at most n. We can decompose the p-Sylow subgroup of S_n as an r-fold wreath product of $\mathbb{Z}/p$, and hence decompose $\Gamma_n(X)$ into a union of $ES_n(p) \times_{S_n(p)} X^n$, which generate the Dyer-Lashof operations. □

- Now suppose X is $k-1$ connected, with $H_k(X) \cong \mathbb{Z}/p$, and let $i : S^k \to X$ be a map representing a generator $z \in H_k(X)$.
- Recall the map $g_{n-1} : D_{n-1}(X) \wedge S^k \to D_n(X)$. On homology, we have that

$$(g_{n-1})_* \sigma_k : H_i(D_{n-1}(X); \mathbb{Z}/p) \to H_i(D_n(X); \mathbb{Z}/p)$$

is the same as the map $\alpha = \times z : A_{n-1}(X) \to A_n(X)$.

Theorem (Theorem 2.5)

We assume that k is even if p is odd. Then $(g_{n-1})_*$ is a mono, and iso for $i < kn + \frac{2p-3}{p} n$.

Proof.

- By the above discussion, it's enough to consider multiplication by z. α is clearly monomorphic since $H_*(Q(X))$ is a free graded algebra. So we must show it's an epimorphism in the right range.
- Suppose we have a monomial of height n, $x = \prod Q^{I_i} x_i$.
- The proof is a counting argument: If $deg(x_i) > k$, then we have that $deg(x) > nk + \frac{n}{2}$.
- This implies that if $deg(x)$ is less than this bound, then at least one of the x_i has degree k and must be z.

□

Theorem (Theorem 3.1)

D_{M_k} has the same mod p homotopy type as a wedge of $H\mathbb{Z}/p$.

- Since we will be considering elements of order p, we should accordingly consider $M_k = S^k \bigcup_p e^{k+1}$, and also $D_\pi(X)$, where π is a cyclic p-group.
- Recall that $H^n(B\pi, \mathbb{Z}/p) \cong \mathbb{Z}/p$ and is generated by w_1^n. Let $e_i \in H_i(B\pi, \mathbb{Z}/p)$ be the dual class.
- By our previous discussion, if $\{x_i\}$ is a basis for $\tilde{H}_*(X)$, then a basis for $\tilde{H}_*(D_\pi(X))$ is given by monomials of height n,

$$e_i \otimes x_j^p \text{ and } e_0 \otimes (x_{j_1} \otimes \cdots \otimes x_{j_p})$$

- Our goal is to show that $H_*(D_{M_k}, \mathbb{Z}/p)$ is a free $\mathcal{A}_p$ algebra. To do so, we will look at the action of $\mathcal{A}_p$ on $H^*(D_{M_k}, \mathbb{Z}/p)$ (as coalgebras over $\mathcal{A}_p$).

- Given a class $u \in H^p(X)$, we can form the external reduced powers $P^{(r)}(u) \in H^{p^r q}(D_{p^r,p}(X) = D_\pi \circ \cdots \circ D_\pi(X))$.
- If $x \in H_k(M_k)$ is the dual of $u \in H^k(M_k)$, then $P^{(r)}(u)$ is the dual of x^{p^r}.
- Since we understand the action of $\mathcal{A}_p^*$ on $H_k(M_k)$ well, we can exploit this to show that the action of $\mathcal{A}_p$ (using the Milnor basis) on $P^{(r)}(u)$ is nontrivial.
- This means that the coalgebra map $\Phi : \mathcal{A}_p \to H^*(D_{M_k})$ is a monomorphism, which implies that the map of algebras is a monomorphism, which implies that $H_*(D_{M_k})$ is a free $\mathcal{A}_p$ algebra.

Theorem (Theorem 4.1)

Let $\alpha \in \pi_k(\mathbb{S})_p$ be of order p. Then for any integer n and any $\gamma \in \pi_t(\mathbb{S})_p$ such that $0 < t < \lfloor \frac{2p-3}{p} n \rfloor - 1$, we have that $\gamma\alpha^n = 0$.

Corollary (Corollary 4.2)

Let $\alpha \in \pi_k(\mathbb{S})_p$ be of order p, and let n be the smallest integer n such that $0 < k < \lfloor \frac{2p-3}{p} n \rfloor - 1$, we have that $\alpha^{n+1} = 0$.

Proof.

We may assume k is even if p is odd. Then suppose α is represented by a map $f : S^{k+N} \to S^N$. Since it is of order p, f extends to a map $\tilde{f} : S^{k+N} \bigcup_p e^{k+N+1} \to S^N$.

$$\begin{array}{ccc}
D_n^{(r)}(S^{k+N} \bigcup_p e^{k+N+1}) & \xrightarrow{D_n^{(r)}(\tilde{f})} & D_n^{(r)}(S^N) \\
\uparrow & & \downarrow \\
D_n^{(r)}(S^{k+N}) & \xrightarrow{D_n^{(r)}(f)} & D_n^{(r)}(S^N) \\
\uparrow & & \downarrow r \\
D_n(S^{k+N}) = S^{n(k+N)} & \xrightarrow{f^{(n)}} & S^{nN}
\end{array}$$

Proof.

- We can choose r to be a retraction, and we can take r and N large enough so that $D_n^{(r)}(S^{k+N} \bigcup_p e^{k+N+1}))$ is mod p stably homotopy equivalent to the wedge of $H\mathbb{Z}/p$ up to dimension $n(k+N) + \frac{2p-3}{p}n$.
- Then for any map $g : S^{n(k+N)+i} \to S^{n(k+N)}$ with $0 < i < \lfloor \frac{2p-3}{p} n \rfloor - 1$, this cannot hit anything in the range, and so the composite is zero.

□

Remark

Note that this bound is not sharp.

- To prove the theorem for any element $\alpha \in \pi_k(\mathbb{S})$, we first observe it is of order p^m. Since the extended n-th power construction is functorial, we know that $D_n(p^m\alpha)$ and also

$$D_n(p^m\alpha^r) \simeq D_n(p^m\iota_k)D_n(\alpha^r)$$

are nullhomotopic.

- So we investigate how the extended nth power construction acts on multiplication of the identity map $\iota_k : S^k \to S^k$. In other words, we would like to understand $D_n(p^m\iota_k)$.

Theorem (Theorem 5.1)

For any n and m, the map ranging over partitions of n of length m

$$f = \vee(D_{s_1}(A_1) \wedge \cdots \wedge D_{s_m}(A_m)) \to D_n(\vee A_i)$$

is a homotopy equivalence.

Corollary (Corollary 5.2)

Given $\vee g_i : \vee A_i \to B$*, then*

$$D_n(\vee g_i)f \sim \vee(\mu(D_{s_1}(g_1) \wedge \cdots \wedge D_{s_m}(g_m))$$

- Taking $A_i = B = S^k$, letting ι_k be the identity map, $\pi : \vee S^k \to S^k$ the natural projection and $\Phi : S^k \to \vee S^k$ the comultiplication map.
- Then $m\iota_k = \pi\Phi$. Hence, applying $D_n(-)$, we have

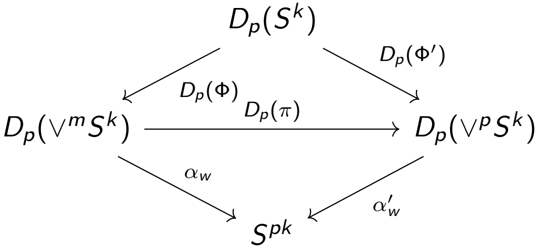
$$\begin{aligned} D_n(m\iota_k) &= D_n(\pi\Phi) \\ &= D_n(\pi)D_n(\Phi) \\ &= (\vee\mu(D_{s_1}(g_1) \wedge \cdots \wedge D_{s_m}(g_m))f^{-1}D_n(\Phi) \end{aligned}$$

- Rewriting the formula so that we're indexing over partitions w, we stably obtain the formula

$$D_n(m\iota_k) \sim \sum \mu_w \alpha_w D_n(\Phi)$$

- We define a partition by a pair of integers (t_i, d_i), where t_i is a sequence of increasing integers with $t_1 = 0$ satisfying $\sum t_i d_i = n$, with $\sum d_i = m$ the multiplicity.
- Setting $n = p$, and $m \equiv 0 (\text{mod } p)$, we will show that there are at least two homotopy classes of maps for $\mu_w \alpha_w D_n(\Phi)$.
- Observe that S_m acts on the set of partitions of n of length m. Note that under this action, $\mu_w \alpha_w D_n(\Phi) \simeq \mu_{\theta^*(w)} \alpha_{\theta^*(w)} D_n(\Phi)$.
- Furthermore, note that the size of the orbit set is $\frac{m!}{\prod(d_i!)}$.
- The first class corresponds to the partition $d_1 = m - p, d_2 = p, d_3 = d_4 \cdots = 0$. The others correspond to partitions with $d_1 > m - p, d_2 < p, \cdots d_m < p$.

- We concern ourselves with the partition $d_1 = m - p, d_2 = p, d_3 = d_4 \cdots = 0$, which corresponds to $w = (0, \dots, 0, 1, \dots, 1)$. Then α_w is a map from $D_p(\vee^m S^k) \to S^{pk}$. This is homotopic to the map $D_p(\vee^p S^k) \to S^{pk}$ via $D_p(\pi)$, where π shrinks the first $m - p$ spheres.



We set $h_p = \alpha_w D_p(\Phi)$ for w a partition as above, up to S_m action.

Theorem (Theorem 5.6)

Letting $j : S^{pk} \to D_p(S^k)$, then we have shown that stably,

$$D_p(m\iota_k) \sim p^r g + \binom{m}{p} jh_p$$

In the case $p = 2$, we see that

$$D_2(m\iota_k) \sim m\iota_{D_2(S^k)} + \binom{m}{2} jh_2$$

And in the case $m = p$, we also have that

$$D_p(p\iota_k) \sim p\iota_{D_p(S^k)} + jh_p$$

In these cases there are only two possible partitions up to S_m action.

- We have now reduced to understanding the map h_p. In particular, we would like to understand the action of the dual Steenrod algebra on h_p.
- In particular, we will show that the action is non-trivial in a certain range.
- To do so, we set $m = p$ and make use of the formula

$$D_p(\iota_k) \sim p\iota_{D_p(S^k)} + jh_p$$

- The action on h_p is non-trivial iff jh_p is nontrivial.
- Furthermore, the action on $D_p(p\iota_k)$ is non-trivial, but the action on $p\iota_{D_p(S^k)}$ is trivial.

Theorem (Theorem 6.5)

$D_2(S^k)$ is stably homotopy equivalent to $\mathbb{R}P_+^{(r)}$ iff $k \equiv 0 (mod\ 2^{\Phi(r)})$, where $\Phi(r) = \#\{i | 0 < i \leq r, i \equiv 0, 1, 2, \text{ or } 4 \bmod 8\}$

Then by work of Kahn-Priddy, since h_p has a nontrivial action of the dual Steenrod algebra, the adjoint of h_p sends a generator in the homology of $\mathbb{R}P_+^r$ to the image of a generator under the map $\mathbb{R}P_+^r \to QS^0$. Under this condition, there exists a splitting $QS^0 \to Q(\mathbb{R}P_+^r)$.

Theorem (Kahn-Priddy)

This induces an epimorphism in homotopy groups for $0 < i < r$:

$$(h_2)_* : \pi_i(\mathbb{R}P^{(r)})_p \to \pi_i(S^0)_p$$

Theorem (Theorem 6.5)

Similarly, $D_\pi(S^k)$ is stably homotopy equivalent to $B\pi_+^{(2r+1)}$ iff $k \equiv 0 (mod\ p^{\lfloor \frac{r}{p-1} \rfloor})$.

Theorem (Kahn-Priddy)

This induces an epimorphism in homotopy groups for $0 < i < r$:

$$(h_p)_* : \pi_i(B\pi^{(r)})_p \to \pi_i(S^0)_p$$

Combining the work of the previous sections, we obtain the following theorem (for $p = 2$):

Theorem (Theorem 8.1)

Let $\alpha \in \pi_k(\mathbb{S})$ be an element of order 2^m and k even. Given any integer n, let r be the maximal integer such that $nk \equiv 0 (mod\ 2^{\Phi(r)})$. Then for any $\beta \in \pi_i(\mathbb{S})$ for $0 < i < r$, we have

$$2^{m-1}(\alpha^{2n}\beta + 2\gamma) = 0$$

for some $\gamma \in \pi_*(\mathbb{S})$.

Proof.

- *Observe that* $D_2(S^{nk}) \simeq \Sigma^{nk}(\mathbb{R}P_+) \simeq S^{2nk} \vee \Sigma^{2nk}\mathbb{R}P$. *Since* k *is even, the* Sq^2 *on* h_2 *is nontrivial.*
- *Hence* $(h_2)_* : \pi_{i+2nk}(D_2(S^n k)) \to \pi_{i+2nk}(S^n k)$ *is an epimorphism for* $0 < i < r$.
- *So we may choose a* $\tilde{\beta}$ *such that* $h_2(\tilde{\beta}) = \beta$.
- *Now, consider* α^n. *This also has order* 2^m, *hence*

$$RD_2(\alpha^n)D_2(2^m) : D_2(S^{nk}) \to D_2(S^{nk}) \to D_2(S^0) \to S^0$$

is nullhomotopic.

Proof.

Therefore, we have that

$$\begin{aligned} 0 &\sim RD_2(\alpha^n)(2^m \iota_{D_2(S^k)} + \binom{2^m}{2} jh_2) \\ &\sim 2^m RD_2(\alpha^n) + 2^{m-1}(2^m - 1) RD_2(\alpha^n) jh_2) \\ &\sim 2^m RD_2(\alpha^n) + 2^{m-1}(2^m - 1)\alpha^{2n} h_2) \end{aligned}$$

Composing with $\tilde{\beta}$, we then have

$$2^m RD_2(\alpha^n)\tilde{\beta} + 2^{m-1}(2^m - 1)\alpha^{2n}\beta \sim 0$$

We set $\gamma = RD_2(\alpha^n)\tilde{\beta}$. □

Corollary (Corollary 8.2)

Any element in the 2-primary positive stem of the stable homotopy groups of spheres is nilpotent.

Proof.

- It is enough to prove this for $\alpha \in \pi_k(\mathbb{S})$ be an element of order 2^m and k even.
- We take $nk \equiv 0 (\text{mod } 2^{\Phi(k+1)})$. Then we may take $\alpha = \beta$.
- Hence we have $2^{m-1}(\alpha^{2n+1} + 2\gamma) \sim 0$.
- Composing with α, since it is of order 2^m, we then obtain that $2^{m-1}(\alpha^{2n+2}) \sim 0$. Iterating this process yields the result.
- The bound on the exponent is roughly $2^{\lfloor \frac{k+1}{2} \rfloor}$

□

For p odd, we have similar results:

Theorem (Theorem 8.3)

Let $\alpha \in \pi_k(\mathbb{S})$ be an element of order p^m. Given any integer n, let r be the maximal integer such that $nk \equiv 0(mod\ p^{\lfloor \frac{r}{p-1} \rfloor})$. Then for any $\beta \in \pi_i(\mathbb{S})$ for $0 < i < 2r$, we have

$$p^{m-1}(\alpha^{pn}\beta + p\gamma) = 0$$

for some $\gamma \in \pi_*(\mathbb{S})$.

Corollary (Corollary 8.4)

Any element in the p-primary positive stem of the stable homotopy groups of spheres is nilpotent.

M G Barratt and Peter J Eccles.
Γ+-Structures—I: a free group functor for stable homotopy theory.
Topology, 13(1):25–45, March 1974.

Robert R Bruner, J Peter May, James E McClure, and Mark Steinberger.
H∞ Ring Spectra and their Applications.
Springer, Berlin, Heidelberg, 1986.

Eldon Dyer and R K Lashof.
Homology of iterated loop spaces.
Amer. J. Math., 84(1):35–88, 1962.

D S Kahn and S B Priddy.
Applications of the transfer to stable homotopy theory.
Bull. Am. Math. Soc., 1972.

Akhil Mathew, Niko Naumann, and Justin Noel.
On a nilpotence conjecture of J.P. may.
March 2014.

Goro Nishida.
The nilpotency of elements of the stable homotopy groups of spheres.
J. Math. Soc. Japan, 25(4):707–732, October 1973.

Douglas C Ravenel.
Nilpotence and Periodicity in Stable Homotopy Theory.
Princeton University Press, 1992.