An Overview of Algebraic Topology

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Outline

Topological Spaces
- What are they?
- How do we build them?
- When are they the same or different?

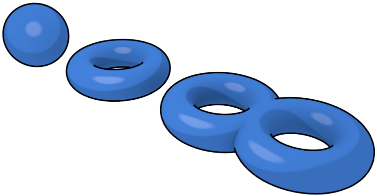
Algebraic Topology
- Homotopy
- Fundamental Group
- Higher Homotopy Groups

What is a topological space?

- Working definition: A set X with a family of subsets τ satisfying certain axioms (called a topology on X). The elements of τ are the open sets.
  1. The empty set and X belong in τ.
  2. Any union of members in τ belong in τ.
  3. The intersection of a finite number of members in τ of belong in τ.
- Most things are topological spaces.
- We care about topological spaces with natural topologies.

Example (Surfaces)

A **surface** is a topological space that locally looks like $\mathbb{R}^2$.



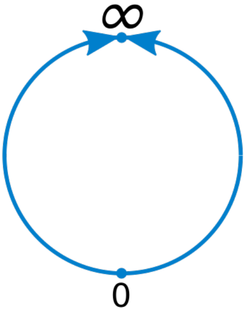
Source: laerne.github.io

Example (Manifolds)

An *n*-**manifold** is a topological space that locally looks like $\mathbb{R}^n$.

Example (Spheres)

An n-**sphere** is the one-point compactification of $\mathbb{R}^n$. We write it as S^n.

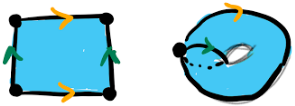


Source: Wikipedia

Building Topological Spaces

- Abstract toplogical spaces are sometimes hard to get a handle on, so we would like to model them with combinatorial objects, called CW complexes.
- To build a CW complex, you start with a set of points, which is called the 0-skeleton.
- Next, you glue in 1-cells (copies of D^1) to the 0-skeleton, such that the boundary of each D^1 is in the boundary. This forms the 1-skeleton.
- You repeat this process, gluing in n-cells (copies of D^n) such that the boundary of each D^n lies inside the $(n-1)$-skeleton.

Examples of CW complexes

- 2-sphere: 
- 2-sphere: 
- 2-Torus: 

Putting CW structures on topological spaces

Theorem (CW approximation theorem)

For every topological space X, there is a CW complex Z and a weak homotopy equivalence $Z \to X$.

When are they the same?

- We almost never have strict equality. So we must choose a perspective of equality to work with.
  - **Homeomorphism**.
  - **Homotopy equivalence**.
  - **Weak homotopy equivalence**.

Definition (homeomorphism)

A map $f : X \to Y$ is a **homeomorphism** if f is bijective continuous map and has a continuous inverse $g : Y \to X$.

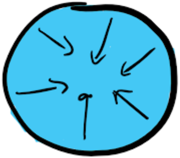
Source: Wikipedia

The coffee cup and donut are homeomorphic.

Definition (homotopy equivalence)

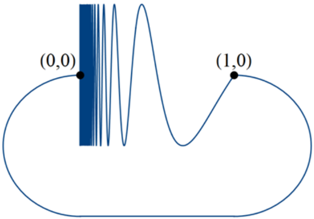
A map $f : X \to Y$ is a **homotopy equivalence** if f is continuous and has a continuous homotopy inverse $g : Y \to X$.

The unit ball is homotopy equivalent, but not homeomorphic, to the point.



Definition (weak homotopy equivalence)

A map $f : X \to Y$ is a **weak homotopy equivalence** if f induces bijections on π_0 and isomorphisms on all homotopy groups.



Source: Math Stackexchange

The Warsaw circle is weakly homotopy equivalent, but not homotopy equivalent, to the point.

Comparison of perspectives

Proposition

Homeomorphism $\Rightarrow$ Homotopy equivalence $\Rightarrow$ Weak homotopy equivalence.

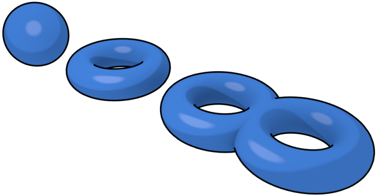
When can we go the other way?

Theorem (Whitehead's theorem)

If $f : X \to Y$ is a weak homotopy equivalence of CW complexes, then f is a homotopy equivalence.

When are they different?

- It's somehow hard to determine whether or not two spaces are the same. It's much easier to tell spaces apart using tools called **invariants**. These invariants depend on your choice of perspective.



Source: laerne.github.io

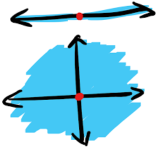
Connectedness

Definition (Connectedness)

A space is **connected** if it cannot be written as the disjoint union of two open sets.

Example

$\mathbb{R} - \{0\}$ is not connected, but $\mathbb{R}^n - \{0\}$ is for $n \geq 2$.



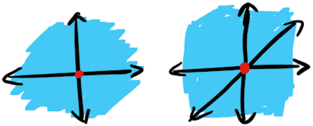
Simple-connectedness

Definition (Simple-connectedness)

A space X is **simply connected** if it is path connected and any loop in X can be contracted to a point.

Example

$\mathbb{R}^2 - \{0\}$ is not simply-connected, but $\mathbb{R}^n - \{0\}$ is for $n \geq 3$.



- Connectedness and simple-connectedness are a manifestation of counting the number of 0 and 1-dimensional “holes” in a topological space.
- We can generalize this notion to an algebraic invariant called **homology**.
- This is how we can tell $\mathbb{R}^n \ncong \mathbb{R}^m$ for $n \neq m$.
- It is much easier to calculate things algebraically, rather than rely on geometry.
- Some other useful invariants are **cohomology** and **homotopy groups**.

Homotopy

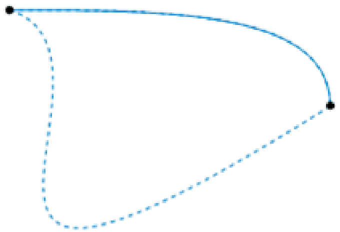
Definition (homotopy of maps)

A **homotopy** between two continuous maps $f, g : X \to Y$ is a continuous function $H : X \times [0, 1] \to Y$ such that for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. We write $f \simeq g$.

Proposition

Homotopy defines an equivalence relation on maps from $X \to Y$.

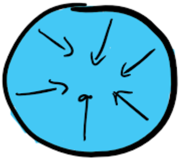
Homotopy



Source: Wikipedia

Definition (homotopy equivalence)

A continuous map $f : X \to Y$ is a **homotopy equivalence** if there exists a continuous map $g : Y \to X$ such that $f \circ g \simeq Id_Y$ and $g \circ f \simeq Id_X$. g is called a homotopy inverse of f.

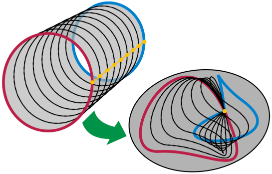


Fundamental Group

Let us now assume that X is path-connected.

Proposition

The set of loops on X with a fixed base point up to homotopy form a group, where the multiplication is concatenation.

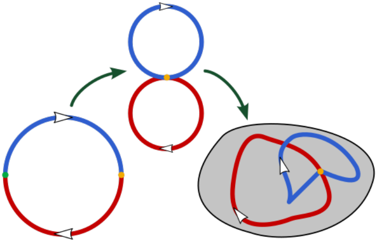


Source: Wikipedia

Fundamental Group

Proposition

The set of homotopy classes of based continuous maps $f : S^1 \to X$ form a group, denoted $\pi_1(X)$.



Source: Wikipedia

Example

If X is contractible, $\pi_1(X) = 0$.

Example

$\pi_1(S^1) \cong \mathbb{Z}$.

This comes from a covering space calculation.

Example

$\pi_1(S^n) \cong 0$ for $n \geq 2$.

Higher homotopy groups

Proposition

The set of homotopy classes of continuous based maps $f : S^n \to X$ form a group, denoted $\pi_n(X)$

There are lots of calculational tools:

- Long exact sequence of a fibration
- Spectral sequences
- Hurewicz theorem
- Blakers-Massey theorem

Higher homotopy groups of spheres

	S^0	S^1	S^2	S^3	S^4	S^5	S^6	S^7	S^8
π_1	0	$\mathbb{Z}$	0	0	0	0	0	0	0
π_2	0	0	$\mathbb{Z}$	0	0	0	0	0	0
π_3	0	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0
π_4	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0
π_5	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
π_6	0	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
π_7	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
π_8	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
π_9	0	0	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
π_{10}	0	0	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_2$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$
π_{11}	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\mathbb{Z}_{24}$
π_{12}	0	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	0	0
π_{13}	0	0	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_2$	0

Source: HoTT book

Freudenthal Suspension Theorem

- This is not a coincidence!

Theorem (Corollary of Freudenthal Suspension Theorem)

For $n \geq k+2$, there is an isomorphism

$$\pi_{k+n}(S^n) \cong \pi_{k+n+1}(S^{n+1})$$

The general theorem says that for fixed k, there is stabilization for highly-connected spaces. We can make spaces highly connected via suspension.

Stable homotopy theory

Definition (stable homotopy groups of spheres)

The k-th stable homotopy group of spheres, $\pi_k^S(S)$, is $\pi_{k+n}(S^n)$ for $n \geq k+2$.

- This is an algebraic phenomenon, and one might wonder if there is a corresponding topological/geometric concept.
- Recall that homotopy groups of X are homotopy classes of maps from $S^n \to X$. Is there a corresponding notion for stable homotopy groups?
- The answer is **yes**!
- This leads to the notion of spectra, which is the stable version of a space, and to stable homotopy theory.

Stable homotopy theory

- Working definition: A spectrum is a sequence of spaces X_n with structure maps $\Sigma X \to X_{n+1}$.
- Given a space X, you can obtain the suspension spectrum $\Sigma^\infty X$ with identities as the structure maps.
- For example, the sphere spectrum $\mathbb{S}$ is the suspension spectrum of the sphere.

- The k-th stable homotopy groups of a space X are homotopy classes of maps from (the k-shifted) sphere spectrum $\mathbb{S}$ to the suspension spectrum $\Sigma^\infty X$.

$$\pi_k^S(X) = [\Sigma^k \mathbb{S}, \Sigma^\infty X]_{\mathsf{Sp}}$$

- We can do the same thing with **generalized cohomology theories**, which are other algebraic invariants.

$$E^n(X) \cong [X, E_n]_{\mathsf{Top}}$$

Summary

- We would like to understand when two topological spaces are the same or different. This depends on our choice of perspective.
- In particular, we would like to compute invariants that can help us answer this question. We use geometric, combinatorial, and algebraic tools to do so.
- Studying these invariants often leads to fascinating new patterns, which in turn brings us new geometric insights like stable phenomena.