The Periodicity Theorem, Part I (HHR Section 9.1)

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The Periodicity Theorem

Theorem (9.21)

Let $G = C_8$, and

$$D = (N_{C_2}^{C_8}\overline{\partial}_4^{C_2})(N_{C_4}^{C_8}\overline{\partial}_2^{C_4})(\overline{\partial}_1^{C_8}) \in \pi_{19\rho_G}^G MU^{((G))}$$

Then multiplication by $(\Delta^G)^{16} := u_{2\rho_G}^{16}(\overline{\partial}_1^{C_8})^{32}$ gives an isomorphism

$$\pi_*(D^{-1}MU^{((G))})^{hG} \to \pi_{*+256}(D^{-1}MU^{((G))})^{hG}$$

Periodicity Theorem Overview

- We will use the $RO(G)$-graded slice spectral sequence to compute $\pi_\star^G(MU^{((G))})$.
  - The E_2 page is given by Proposition 9.7.
  - Differentials are given by Theorem 9.9.
- We then show that a certain class $\overline{\partial}_k u^{2^k}$ is a permanent cycle (Corollary 9.13).
- This implies that a class $(\Delta^G)^{2^{g/2}}$ is a permanent cycle in the $RO(G)$-graded slice spectral sequence for $\pi_\star^G(D^{-1}MU^{((G))})$.
- This class restricts to a unit in $\pi_\star^u(D^{-1}MU^{((G))})$, and hence multiplication by $(\Delta^G)^{2^{g/2}}$ gives us the Periodicity Theorem.

The $RO(G)$-graded slice spectral sequence

Theorem

We have the $RO(G)$-graded slice spectral sequence for $MU^{((G))}$:

$$E_2^{s,t} := \pi_{t-s}^G P_{\dim t}^{\dim t} X \Rightarrow \pi_{t-s}^G MU^{((G))}$$

with differentials $d_r : E_2^{s,t} \to E_2^{s+r,t+(r-1)}$

Where $t \in -2m\sigma + \mathbb{Z}$

Recall that $P_0^0 MU^{((G))} \cong H\underline{\mathbb{Z}}_{(2)}$

Let $\sigma = \sigma_G$ denote the real sign representation of G, and recall that in Definition 3.12, we defined an element

$$u = u_{2\sigma} \in \pi^G_{2-2\sigma} H\underline{\mathbb{Z}}_{(2)} = E_2^{0,2-2\sigma}$$

Corresponding to a preferred generator of $\pi_2(H\underline{\mathbb{Z}}_{(2)} \wedge S^{2\sigma})$. We will study the elements

$$u^m \in E_2^{0,2m-2m\sigma}$$

in the $RO(G)$-graded slice spectral sequence for $\pi^G_\star(MU^{((G))})$.

Consider the $\mathbb{Z} \times RO(G)$-graded ring

$$\mathbb{Z}_{(2)}[a, f_i, u]/(2a, 2f_i)$$

with $|a| = (1, 1 - \sigma)$, $|f_i| = (i(g-1), ig)$, and $|u| = (0, 2 - 2\sigma)$

Proposition (9.7)

The map

$$\mathbb{Z}_{(2)}[a, f_i, u]/(2a, 2f_i) \to \bigoplus_{\substack{s,k \geq 0 \\ t \in * - k\sigma}} E_2^{s,t}$$

is an isomorphism in the range

$$s \geq (g-1)((t-s) - (k - k\sigma))$$

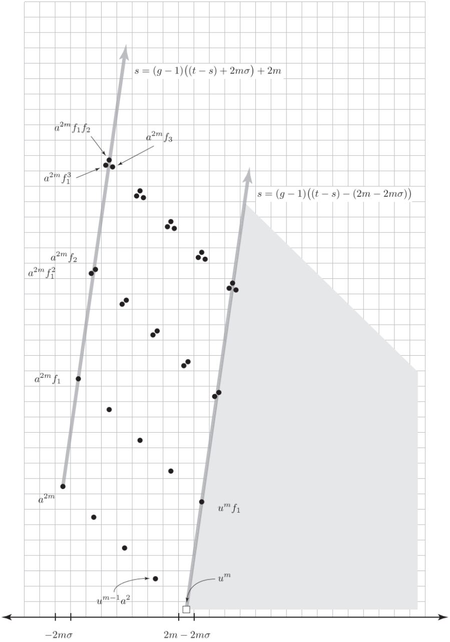

FIGURE 2. The slice spectral sequence for $\pi^G_{-2m\sigma+*} MU^{((G))}$

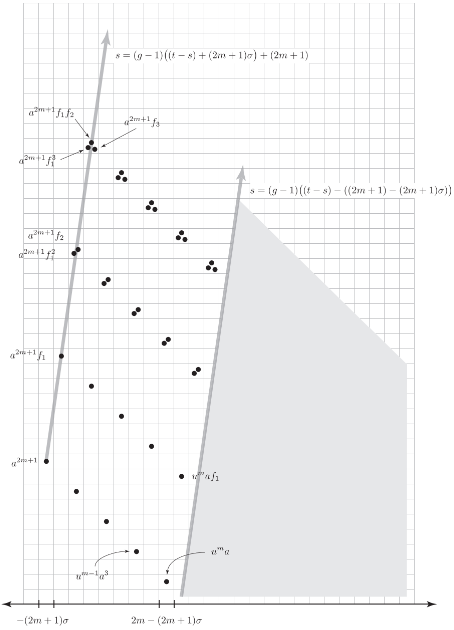

FIGURE 3. The slice spectral sequence for $\pi^G_{-(2m+1)\sigma+*}MU^{((G))}$

The map

$$\mathbb{Z}_{(2)}[a, f_i, u]/(2a, 2f_i) \to \bigoplus_{\substack{s,k \geq 0 \\ t \in * - k\sigma}} E_2^{s,t}$$

is given by

$$f_i \mapsto a_{\rho}^{i} N\overline{r}_i \in \pi_i^G P_{ig}^{ig} MU^{((G))}$$

$$a \mapsto a_{\sigma} \in \pi_{-\sigma}^G P_0^0 MU^{((G))}$$

$$u \mapsto u \in \pi_{2-2\sigma}^G P_0^0 MU^{((G))}$$

Proposition

In the slice spectral sequence, we have a vanishing line

$$s = (g-1)((t-s)+k\sigma)+k$$

Proof.

This follows by setting $t' = \dim t$ so that $t = t' + (k - k\sigma)$. We then have that

$$E_2^{s,t} = \pi^G_{t'-s+k} S^{k\sigma} \wedge P^{t'}_{t'} MU^{((G))}$$

Note that $S^{k\sigma} \wedge P^{t'}_{t'} MU^{((G))} \geq t'$, so Proposition 4.40 tells us that this group vanishes if $t' - s + k < \lfloor t'/g \rfloor$.

Hence if $s > (g-1)((t-s)+k\sigma)+k$ □

By the Slice Theorem (Theorem 6.1 cf Theorem 1.13), $P_{t'}^{t'} MU^{((G))}$ is contractible unless t' is even.

If t' is even, then $P_{t'}^{t'} MU^{((G))} \simeq \bigvee H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S}$, where $\widehat{S}$ is a slice cell of dimension t'.

We compute $E_2^{s,t} = \pi^G_{t'-s+k} S^{k\sigma} \wedge P_{t'}^{t'} MU^{((G))}$ by considering the two cases for $\widehat{S}$:

1. Either $\widehat{S} = G_+ \wedge_H S^{\ell' \rho_H}$ is an induced slice cell,
2. Or $\widehat{S} = S^{\ell \rho_G}$ is an non-induced slice cell.

We will see that in the range $s \geq (g-1)((t-s)-(k-k\sigma))$, the homotopy groups coming from the induced slice cells vanish, and only the non-induced slice cells contribute.

Case 1

If $\widehat{S} = G_+ \wedge_H S^{\ell' \rho_H}$ is an induced slice cell, then we are interested in computing $\pi^G_{t'-s+k}$ of

$$S^{k\sigma} \wedge H\underline{\mathbb{Z}}_{(2)} \wedge G_+ \wedge_H S^{\ell' \rho_H}$$

Since the restriction of σ to any proper subgroup is trivial, then this is homotopic to

$$G_+ \wedge_H (S^{k\sigma} \wedge H\underline{\mathbb{Z}}_{(2)} \wedge S^{\ell' \rho_H}) \simeq G_+ \wedge_H (S^{k} \wedge H\underline{\mathbb{Z}}_{(2)} \wedge S^{\ell' \rho_H})$$

Hence we are interested in computing $\pi^H_{t'-s}(H\underline{\mathbb{Z}}_{(2)} \wedge S^{\ell' \rho_H})$

Case 1

By Proposition 4.40, $\pi^H_{t'-s}(H\underline{\mathbb{Z}}_{(2)} \wedge S^{\ell'\rho_H})$ vanishes if

$$t' - s < \ell' = t'/h \qquad (h = |H|)$$

so it in particular vanishes for $t' - s < t'/g$. Equivalently, it vanishes for

$$s \geq (g-1)((t-s) - (k - k\sigma))$$

Case 2

If $\widehat{S} = S^{\ell\rho_G}$ is an non-induced slice cell, then we are interested in computing

$$\pi_j^G(S^{k\sigma} \wedge H\underline{\mathbb{Z}}_{(2)} \wedge S^{\ell\rho_G})$$

for $j \leq \ell + k$ and $\ell, k \geq 0$.

Lemma (9.1)

$$\pi_j^G(S^{k\sigma} \wedge H\underline{\mathbb{Z}}_{(2)} \wedge S^{\ell\rho_G}) \cong$$

$$\begin{cases} 0 & \text{if } (j-\ell) < 0 \text{ or } (j-\ell) \text{ odd} \\ \mathbb{Z}/2 \cdot \{a_\rho^\ell a_\sigma^{k-2m} u_{2\sigma}^m\} & \text{if } (j-\ell) = 2m \geq 0 \text{ and } \ell > 0 \\ \mathbb{Z}_{(2)} \cdot \{u_{2\sigma}^m\} & \text{if } (j-\ell) = 2m \geq 0 \text{ and } \ell = 0 \end{cases}$$

Proof.

We write $S^{k\sigma} \wedge S^{\ell\rho_G} = S^{(k+\ell)\sigma} \wedge S^{\ell} \wedge S^{\ell(\rho_G - \sigma - 1)}$ and consider the multiplication map

$$a^{\ell}_{\bar{\rho}-\sigma} : \pi^G_j H\underline{\mathbb{Z}}_{(2)} \wedge S^{(k+\ell)\sigma} \wedge S^{\ell} \to \pi^G_j H\underline{\mathbb{Z}}_{(2)} \wedge S^{k\sigma} \wedge S^{\ell\rho_G}$$

We claim that this map is an isomorphism for $j \leq \ell + k$ and $\ell, k \geq 0$.

If $\ell = 0$, this map is an isomorphism. □

Proof.

For $\ell > 0$, $S^{\ell(\rho_G - \sigma - 1)}$ has one 0-cell, and all other G-cells are induced and in positive dimension.

Since the restriction of σ to every proper subgroup is trivial, it follows that to obtain $S^{k\sigma} \wedge S^{\ell\rho_G}$ from $S^{(k+\ell)\sigma} \wedge S^{\ell}$, one attaches induced G-cells of dimension greater than $(k + 2\ell)$.

Hence $a^{\ell}_{\rho - \sigma}$ is an isomorphism for $j < k + 2\ell$, and hence for $j \leq \ell + k$ since $\ell > 0$.

Therefore, in our range, we are intersted in computing

$$\pi_j^G(H\underline{\mathbb{Z}}_{(2)} \wedge S^{(k+\ell)\sigma} \wedge S^{\ell})$$

Which was done in Proposition 3.16. □

It remains to identify the summand of non-induced slice cells in $MU^{((G))}$. That is, we need the algebra structure as well.

Recall that we have an associative algebra equivalence

$$\bigvee_{k \in \mathbb{Z}} P_k^k MU^{((G))} \simeq H\underline{\mathbb{Z}}_{(2)} \wedge S^0[G \cdot \bar{r}_1, \cdots]$$

We can do so by identifying the summand of non-induced slice cells in each $S^0[G \cdot \bar{r}_i]$ and smashing them together.

Proposition

The associative algebra map

$$S^0[N\bar{r}_1, \cdots] \to S^0[G \cdot \bar{r}_1, \cdots]$$

is the inclusion of the summand of non-induced slice cells.

Proof.

Take the generating inclusion $\bar{r}_i : S^{i\rho_{C_2}} \to S^0[\bar{r}_i]$

We then apply the norm $N_{C_2}^G$ to obtain $N\bar{r}_i : S^{i\rho_G} \to S^0[G \cdot \bar{r}_i]$.

We can then extend it to an associative algebra map $S^0[N\bar{r}_i] \to S^0[G \cdot \bar{r}_i]$, which we claim is the inclusion of the summand of non-induced slice cells.

Recall that

$$S^0[G \cdot \bar{r}_i] \simeq \bigvee_{f : G/C_2 \to \mathbb{N}_0} S^{V_f}$$

□

Proof.

$$S^0[G \cdot \bar{r}_i] \simeq \bigvee_{f: G/C_2 \to \mathbb{N}_0} S^{V_f}$$

Decompose the right hand side over the G-orbits.

Since an indexed wedge over a G-orbit is induced from the stabilizer of any element of the orbit, the summand of non-induced slice cells consists of those f which are constant.

If f is the constant function n, then $V_f = n\rho_G$, hence the summand of non-induced slice cells is

$$\bigvee_{\underline{n}} S^{n\rho_G}$$

Consider the $\mathbb{Z} \times RO(G)$-graded ring

$$\mathbb{Z}_{(2)}[a, f_i, u]/(2a, 2f_i)$$

with $|a| = (1, 1 - \sigma)$, $|f_i| = (i(g-1), ig)$, and $|u| = (0, 2 - 2\sigma)$

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The map

$$\mathbb{Z}_{(2)}[a, f_i, u]/(2a, 2f_i) \to \bigoplus_{\substack{s,k \geq 0 \\ t \in * - k\sigma}} E_2^{s,t}$$

is an isomorphism in the range

$$s \geq (g-1)((t-s) - (k - k\sigma))$$

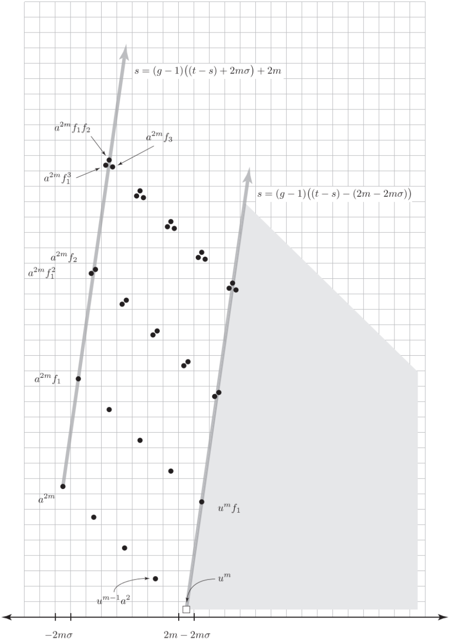

FIGURE 2. The slice spectral sequence for $\pi^G_{-2m\sigma+*}MU^{((G))}$

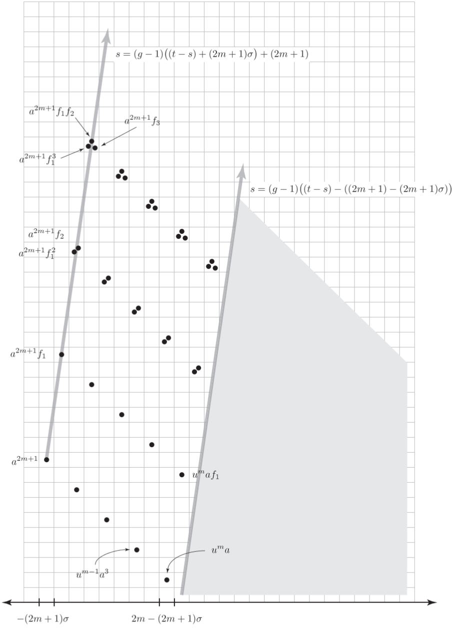

FIGURE 3. The slice spectral sequence for $\pi^G_{-(2m+1)\sigma+*}MU^{((G))}$

By construction, the f_i represent the elements $f_i = a_{\bar{\rho}_G}^{k} N\bar{r}_i \in \pi_i^G MU^{((G))}$, and are hence permanent cycles.

Similarly, a represents the element $a_\sigma \in \pi_{-\theta}^G MU^{((G))}$, and also is a permanent cycle.

Theorem (9.9)

In the slice spectral sequence for $\pi_\star^G(MU^{((G))})$, the differentials $d_i(u^{2^{k-1}})$ are zero for $i < r := 1 + (2^k - 1)g$, and

$$d_r(u^{2^{k-1}}) = a^{2^k} f_{2^k - 1}$$

Note that on the vanishing line

$$s = (g-1)((t-s) + k\sigma) + k$$

is the algebra

$$\mathbb{Z}_{(2)}[a, f_i]/(2a, 2f_i)$$

Recall that in Proposition 5.50, the kernel of the map

$$\Phi^G : H\underline{\mathbb{Z}}_{(2)}[N\bar{r}_1, \cdots] \to \pi_\star^G \Phi^G MU^{((G))} = \pi_* MO[a_\sigma^{\pm 1}]$$

is the ideal $(2, f_1, f_3, f_7, \cdots)$

Hence any non-trivial differentials into the vanishing line must land in this idea.

Proof.

We prove the Slice differential theorem by induction on k. Assume the result for $k' < k$.

In the range $s \geq (g-1)(t-s-(k-k\sigma))$, after resolving the induction differentials, there are two modules over $\mathbb{Z}_{(2)}[f_i]/(2f_i)$: one generated by a^k, which is free over the quotient ring

$$\mathbb{Z}/2[f_i]/(f_1, f_3, \cdots, f_{2^{k-1}-1})$$

and one generated by $u^{2^{k-1}}$.

Since the differential must land in $(2, f_1, f_3, f_7, \cdots)$, for degree reasons, the only possible differential on $u^{2^{k-1}}$ is the one asserted by the theorem. We must show that $u^{2^{k-1}}$ does not survive the spectral sequence. □

Proof.

It suffices to do so after inverting a. Recall that for $G = C_{2^n}$, up to fibrant replacement,

$$\pi_\star \Phi^G(X) = \pi_\star^G(\tilde{E}P \wedge X) \cong a_\sigma^{-1}\pi_\star^G X$$

Inverting a on the map $\pi_\star^G MU^{((G))} \to \pi_\star^G H\underline{\mathbb{Z}}_{(2)}$ yields the map

$$\pi_* \Phi^G(MU^{((G))}) = \pi_* MO \to \pi_* \Phi^G(H\underline{\mathbb{Z}}_{(2)})$$

By Proposition 7.6, this map is 0 in positive degrees. However, if $u^{2^{k-1}}$ is a permanent cycle, so is $a^{-2^k}u^{2^{k-1}}$, but this would represent a class $b^{2^{k-1}}$ in $\pi_*\Phi^G(H\underline{\mathbb{Z}}_{(2)}) \cong \mathbb{Z}/2[b]$, which is a contradiction. □

Permanent cycles

Write

$$\overline{\partial}_k = N\overline{r}_{2^k-1} \in \pi^G_{(2^k-1)\rho_G} MU^{((G))}$$

Note that $f_{2^k-1} = a_{\overline{\rho}}^{2^k-1}\overline{\partial}_k$.

Also observe that we have the identity

$$f_{2^{k+1}-1}\overline{\partial}_k = a_{\overline{\rho}}^{2^{k+1}-1}\overline{\partial}_{k+1}\overline{\partial}_k = f_{2^k-1}a_{\overline{\rho}}^{2^k}\overline{\partial}_{k+1}$$

$\overline{\partial}_k$ is represented in the $RO(G)$-graded slice spectral sequence by an element also denoted $\overline{\partial}_k \in \pi^G_{(2^k-1)\rho_G} P^{(2^{k-1})g}_{(2^{k-1})g} MU^{((G))}$

Corollary (9.13)

In the $RO(G)$-graded slice spectral sequence for $MU^{((G))}$, the class $\overline{\partial}_k u^{2^k}$ is a permanent cycle.

Proof.

Set $r = 1 + (2^{k+1} - 1)g$. By the Slice differential theorem, the differentials $d_i(\overline{\partial}_k u^{2^k}) = \overline{\partial}_k d_i(u^{2^k})$ are zero for $i < r$. Moreover,

$$d_r(\overline{\partial}_k u^{2^k}) = \overline{\partial}_k a^{2^{k+1}} f_{2^{k+1}-1} = a^{2^{k+1}} f_{2^k-1} a_{\overline{\rho}}^{2^k} \overline{\partial}_k$$

However, setting $r' = 1 + (2^k - 1)g$, note that $r' < r$. We also have

$$d_{r'}(u^{2^{k-1}} a^{2^k} a_{\overline{\rho}}^{2^k} \overline{\partial}_{k+1}) = a^{2^k} f_{2^k-1} a_{\overline{\rho}}^{2^k} \overline{\partial}_{k+1}$$

Therefore, we actually have that $d_r(\overline{\partial}_k u^{2^k}) = 0$. □

Proof.

It remains to show that the higher differentials vanish. We show that they land in a region that is 0 in the E_2 term.

Note that $\overline{\partial}_k u^{2^k} \in \pi^G_{2^k(2-2\sigma)+(2^k-1)\rho_G} P^{(2^{k-1})g}_{(2^{k-1})g} MU^{((G))}$

The differential d_{i+1} decreases $t-s$ degree by 1, and increases t degree by i. Hence we wish to study

$$\pi^G_{2^k(2-2\sigma)+(2^k-1)\rho_G-1} P^{(2^{k-1})g+i}_{(2^{k-1})g+i} MU^{((G))}$$

for $i+1 > r = 1+(2^{k+1}-1)g$. □

Proof.

Equivalently, we wish to study

$$\pi^G_{2^{k+1}-1}(S^{2^{k+1}\sigma} \wedge S^{-(2^k-1)\rho_G} \wedge P^{(2^{k-1})g+i}_{(2^{k-1})g+i} MU^{((G))})$$

We rewrite this as

$$\pi^G_{2^{k+1}-1}(S^{2^{k+1}\sigma} \wedge X_i)$$

Note that $X_i \geq i$, so by Proposition 4.40, $\pi^G_j X_i = 0$ for $j < \lfloor i/g \rfloor$.

Since $S^{2^{k+1}\sigma}$ is (-1)-connected, then $\pi^G_{2^{k+1}-1}(S^{2^{k+1}\sigma} \wedge X_i)$ vanishes for $i \geq 2^{k+1}g$. □

Proof.

For the remaining values of i, since they are strictly between $2^{k+1}g$ and $(2^{k+1}-1)g$, then i is not divisible by g. Since $MU^{((G))}$ is pure, then $P^{(2^{k-1})g+i}_{(2^{k-1})g+i}MU^{((G))})$ is induced from a proper subgroup. Therefore, so is X_i.

We therefore have an equivalence

$$S^{2^{k+1}\sigma} \wedge X_i \simeq S^{2^{k+1}} \wedge X_i$$

Therefore, we have that

$$\pi^G_{2^{k+1}-1}(S^{2^{k+1}\sigma} \wedge X_i) = \pi^G_{2^{k+1}-1}(S^{2^{k+1}} \wedge X_i) = 0$$

since $X_i \geq 0$.

□