Applications of Group Cohomology to 3-Manifolds

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Let G be a topological group.

Is it possible to construct a space X such that $\pi_1(X) \cong G$?

If G is discrete, then **yes!** We can use covering space theory.

We need a simply connected space Y such that G acts freely on Y. Then $Y \xrightarrow{p} X = Y/G$ is a universal covering space, and we have

$$G \cong \pi_1(X)/p_*(\pi_1(Y)) \cong \pi_1(X)$$

We can do even better: For discrete G, we can build a space $K(G,1)$ such that $\pi_n(K(G,1)) \cong \begin{cases} G & n = 1 \\ 0 & \textit{else} \end{cases}$

We just need a contractible space Y such that G acts freely on Y.

How does this generalize to arbitrary topological groups?
One can construct BG, the *classifying space* of G. Unfortunately, this is no longer a $K(G,1)$.

Construction

One construction is the Milnor construction, which constructs EG, a contractible CW complex such that G acts freely.

$$EG = colim_i G^{*i}$$

Then $BG = EG/G$, the quotient space of the G-action.

One can think of this as generalization of covering space theory.

Example

$E\mathbb{Z} = \mathbb{R}$, $B\mathbb{Z} \simeq S^1$

Example

$E\mathbb{Z}/2 = S^\infty$, $B\mathbb{Z}/2 \simeq \mathbb{R}P^\infty$

Example

If G is a discrete group, then $BG \simeq K(G, 1)$.

Example

$BS^1 \simeq \mathbb{C}P^\infty$

Example

$B(G \times H) \simeq BG \times BH$

BG is a nice object to study. It is called the classifying space since

$$[X, BG] = \{\text{Isomorphism classes of principal } G\text{-bundles over } X\}$$

Definition

Recall that a principal G-bundle over X is a fiber bundle $\pi : P \to X$ with fiber G, where G acts on itself by (left) translations.

In particular, G acts freely on P, and we always have a fibration $P \to X \to BG$.

Example

$EG \to BG$ is the universal principal bundle.

Definition

The group cohomology of G is defined to be the cohomology of BG:

$$H^*(G;\mathbb{Z}) := H^*(BG;\mathbb{Z})$$

More generally, given a $\mathbb{Z}$-module M, one can define

Definition

The group cohomology of G with coefficients in M is defined to be the cohomology of BG:

$$H^*(G;M) := H^*(BG;M)$$

Example

$$H^i(\mathbb{Z}/2;\mathbb{Z}) := H^i(\mathbb{R}P^\infty;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \text{ odd } \geq 1 \\ \mathbb{Z}/2 & i \text{ even } \geq 2 \end{cases}$$

Example

$$H^i(\mathbb{Z}/2;\mathbb{F}_2) := H^i(\mathbb{R}P^\infty;\mathbb{F}_2) \cong \mathbb{F}_2[x], \text{ with } |x| = 1$$

What if G acts (linearly) on M in an interesting way?

One answer: Group cohomology with local coefficients. This takes into account the action of $\pi_1(BG) = G$ on M.

A $\mathbb{Z}$-module with G-action is the same as a $\mathbb{Z}G$-module. So when considering the category of modules with G action, one can instead consider the category of $\mathbb{Z}G$-modules.

For example, one defines the cohomology of BG with local coefficients M to be

$$H^*(G; M) := H^*(\mathrm{Hom}_{\mathbb{Z}G}(C_n(EG), M))$$

What if G acts (linearly) on M in an interesting way?

Another answer: Homological algebra:

Since EG is contractible, and G acts freely on EG, then we have that $C_*(EG)$ is a **free resolution** of $\mathbb{Z}$ over $\mathbb{Z}G$.

Therefore, we have that

$$H^*(G; M) \cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, M)$$

This gives us a purely algebraic way of understanding group cohomology.

$G = \mathbb{Z}/n$, $\mathbb{Z}$ and $\mathbb{F}_p$ with trivial action.

Example

$$\cdots \xrightarrow{\cdot(\Sigma g^i)} \mathbb{Z}G \xrightarrow{\cdot(g-1)} \mathbb{Z}G \xrightarrow{\cdot(\Sigma g^i)} \mathbb{Z}G \xrightarrow{\cdot(g-1)} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z}$$

$$H^i(\mathbb{Z}/n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \text{ odd } \geq 1 \\ \mathbb{Z}/n & i \text{ even } \geq 2 \end{cases}$$

Example

$$\cdots \xrightarrow{\cdot(\Sigma g^i)} \mathbb{F}_pG \xrightarrow{\cdot(g-1)} \mathbb{F}_pG \xrightarrow{\cdot(\Sigma g^i)} \mathbb{F}_pG \xrightarrow{\cdot(g-1)} \mathbb{F}_pG \xrightarrow{\epsilon} \mathbb{F}_p$$

$H^*(\mathbb{Z}/p;\mathbb{F}_p) \cong \mathbb{F}_p[x] \otimes \Lambda(y)$, with $|x| = 2$ and $|y| = 1$

Theorem (Künneth Formula)

Let X and Y be topological spaces and F be a field. Then for each integer k we have a natural isomorphism

$$\bigoplus_{i+j=k} H^i(X;F) \otimes H^j(Y;F)) \to H^k(X \times Y;F)$$

Example

If k is a field of characteristic p, and G is $(\mathbb{Z}/p)^n$, then

$$H^*(G,k) = \begin{cases} \mathbb{F}_p[x_1, \dots x_n] & |x_i| = 1,\ p = 2 \\ \mathbb{F}_p[x_1, \dots x_n] \otimes \Lambda(y_1, \dots y_n) & |x_i| = 2, |y_i| = 1,\ p \neq 2 \end{cases}$$

Definition

Given a fibration $F \to X \to B$, we have the Serre spectral sequence:

$$E_2^{p,q} = H^q(B; H^q(F)) \Rightarrow H^{p+q}(X)$$

This spectral sequence is a computational tool whose inputs are $H^*(B)$ and $H^*(F)$. If we can also figure out some additional information (the differentials), then we can compute $H^*(X)$.

Example

A SES of groups $1 \to N \to G \to G/N \to 1$ yields a fibration $BN \to BG \to B(G/N)$. Given a G-module M, the associated spectral sequence is the Lyndon-Hochschild-Serre spectral sequence:

$$E_2^{p,q} = H^q(G/N; H^q(N; M)) \Rightarrow H^{p+q}(G; M)$$

From now on, let G be a finite group.

When does G act freely on S^n?

Proposition

If n is even, then the only non-trivial finite group that can act freely on S^n is $\mathbb{Z}/2$.

Proof.

We have a group homomorphism $\deg : G \to \mathbb{Z}/2$ by taking the degree of the map $S^n \xrightarrow{\cdot g} S^n$.
Since G acts freely, $\cdot g$ is a fixed point free map for nontrivial g.
Therefore, by the hairy ball theorem, $\cdot g$ is homotopic to the antipodal map.
Hence for nontrivial $g \in G$, we have $\deg(g) = -1$. Hence $\deg : G \to \mathbb{Z}/2$ is injective. □

Warm up: Finite Groups acting freely on S^1:

Proposition

$\mathbb{Z}/n$ is the only finite group that acts freely on S^1.

Proof.

If G is a finite group acting freely on S^1, then we have a fiber bundle

$$G \to S^1 \xrightarrow{p} S^1/G$$

However, we know that $S^1/G \cong S^1$. We then have by covering space theory that

$$G \cong \pi_1(S^1/G)/p_*(\pi_1(S^1)) \cong \mathbb{Z}/p_*(\mathbb{Z})$$

□

Let $n > 1$. If G acts freely on S^n, then it again a covering space action, and so we again obtain a fiber bundle

$$G \to S^n \xrightarrow{p} S^n/G$$

S^n/G is a closed manifold. Moreover, note that since $\pi_1(S^n) \cong 0$, we have by covering space theory

$$G \cong \pi_1(S^n/G)$$

How does group cohomology come into the picture?

Definition

A finite group G is **periodic** of period $k > 0$ if $H^i(G;\mathbb{Z}) \cong H^{i+k}(G;\mathbb{Z})$ for all $i \geq 1$, where $\mathbb{Z}$ has trivial G action.

Proposition

If G acts freely on S^n, then G is periodic of period $n+1$.

Proof.

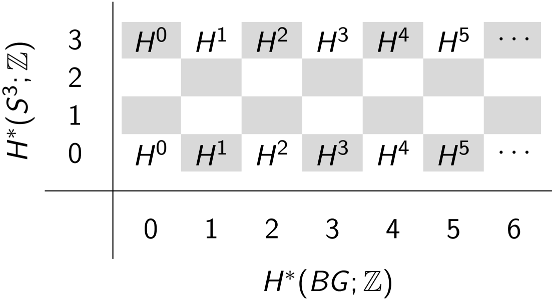
If n is even, this is true, since we saw $H^i(\mathbb{Z}/2;\mathbb{Z})$ is even periodic. So we need to prove the statement for n odd. We consider the fibration

$$S^n \to S^n/G \to BG$$

Note that $\cdot g : S^n \to S^n$ is fixed point free, and hence has degree 1 (and is orientation preserving). So the action of G on $H^*(S^n)$ is trivial. So we now compute the Serre spectral sequence:

$$E_2^{p,q} = H^q(BG; H^q(S^n;\mathbb{Z})) \Rightarrow H^{p+q}(S^n/G;\mathbb{Z})$$

□



The only non-trivial differential is a $d_3 : E_2^{i,3} \to E_2^{i+3+1,0}$.
Furthermore, S^3/G is 3-dimensional, and hence $H^*(S^3/G) \cong 0$ for $* > 3$.
Therefore, $E_\infty^{p,q} = 0$ for $p+q > 3$. For example, the $H^5(G)$ in degree (5,0) must be killed, and so $d_3 : H^1(G) \to H^5(G)$ must be surjective. But it must also be injective, since $H^1(G)$ in degree 4 cannot survive.

Proposition

G is periodic iff all the abelian subgroups of G are cyclic.

Theorem (Suzuki-Zassenhaus)

There are 6 families of periodic groups.

I $\mathbb{Z}/m \rtimes \mathbb{Z}/n$ *with m, n coprime.*

II $\mathbb{Z}/m \rtimes (\mathbb{Z}/n \times Q_{2^k})$ *with m, n, and 2 coprime.*

III $(\mathbb{Z}/m \times \mathbb{Z}/n) \rtimes T_i$ *where m, n, and 6 coprime.*

IV *Groups coming from $TL_2(\mathbb{F}_3) \cong 2S_4$*

V $(\mathbb{Z}/m \rtimes \mathbb{Z}/n) \times SL_2(\mathbb{F}_p)$ *with $m, n, (p^2 - 1)$ coprime, $p \geq 5$.*

VI *Groups coming from $TL_2(\mathbb{F}_p)$* *for $p \geq 5$*

Example

$\mathbb{Z}/p \times \mathbb{Z}/p$ does not act freely on S^n:

$$H^i(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ (\mathbb{Z}/p)^{\frac{i-1}{2}} & i \text{ odd}, \ i \geq 1 \\ (\mathbb{Z}/p)^{\frac{i+2}{2}} & i \text{ even}, \ i \geq 2 \end{cases}$$

One can deduce this using the Kunneth formula to calculate $H^*(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{F}_p) \cong \mathbb{F}_p[x_1, x_2] \otimes \Lambda(y_1, y_2)$. We can then use the universal coefficient theorem to recover integral coefficients.

Warning: Not every periodic group with period 4 acts freely on S^3:

Example

S_3 has period 4, but does not act freely on S^3.

$$H^i(S_3;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2 & i = 2 \bmod 4 \\ \mathbb{Z}/6 & i = 0 \bmod 4 \\ 0 & i \text{ odd} \end{cases}$$

Milnor showed that if a finite group G acts freely on S^n, then every element of order 2 in G is central.

Theorem (Madsen-Thomas-Wall)

A finite group G acts freely on some sphere iff G is periodic and every element of order 2 in G is central.

Example

$\mathbb{Z}/p$ acts freely on S^3.
Consider the unit sphere $S^3 \subseteq \mathbb{C}^2$. Then for any q coprime to p, $\mathbb{Z}/p$ acts by multiplication by

$$\begin{bmatrix} e^{\frac{2\pi i}{p}} & 0 \\ 0 & e^{\frac{2\pi q i}{p}} \end{bmatrix}$$

The quotient manifolds S^3/G are the 3-dimensional lens spaces $L(p; q)$.

Theorem (Suzuki-Zassenhaus)

There are 6 families of periodic groups.

- I $\mathbb{Z}/m \rtimes \mathbb{Z}/n$ *with* m, n *coprime.*
- II $\mathbb{Z}/m \rtimes (\mathbb{Z}/n \times Q_{2^k})$ *with* $m, n,$ *and 2 coprime.*
- III $(\mathbb{Z}/m \times \mathbb{Z}/n) \rtimes T_i$ *where* $m, n,$ *and 6 coprime.*
- IV *Groups coming from* $TL_2(\mathbb{F}_3) \cong 2S_4$
- V $(\mathbb{Z}/m \rtimes \mathbb{Z}/n) \times SL_2(\mathbb{F}_p)$ *with* $m, n, (p^2 - 1)$ *coprime,* $p \geq 5$.
- VI *Groups coming from* $TL_2(\mathbb{F}_p)$ *for* $p \geq 5$

Theorem (Madsen-Thomas-Wall)

A finite group G *acts freely on some sphere iff* G *is periodic and every element of order 2 in* G *is central.*

Theorem (Wolf)

There are 5 families of finite groups that act freely on S^3:

- ***Cyclic case:*** $G \cong \mathbb{Z}/n$
- ***Dihedral case:*** $G \cong \langle x, y \mid xyx^{-1} = y^{-1}, x^{2m} = y^n \rangle$ *for* m, n *coprime, with* $m \geq 1$*,* $n \geq 2$*. For example,* Q_8*.*
- ***Tetrahedral case:*** $G \cong \langle x, y, z \mid (xy)^2 = x^2 = y^2, zxz^{-1} = xy, z^{3^k} = 1 \rangle$ *for* $m, 6$ *coprime, with* $k, m \geq 1$*. For example,* $2A_4$*.*
- ***Octahedral case:*** $G \cong 2S_4$
- ***Icosahedral case:*** $G \cong 2A_5$

And direct products of any of the above groups with a cyclic group of relatively prime order.

Example

In the cyclic case, the S^3/G are Lens spaces.

Example

In the dihedral case, the S^3/G are Prism manifolds.

Example

In the icosahedral case, $S^3/(2A_5)$ is the Poincare homology sphere.

All the 3-manifolds S^3/G arising from those families have finite fundamental group:

$$\pi_1(S^3/G) \cong G$$

Definition

A 3-manifold is spherical if it is of the form

$$M = S^3/G$$

Classifying these was known as the spherical space form problem.

Theorem (Elliptization conjecture)

A 3-manifold with finite fundamental group is a spherical manifold.

This is equivalent to the Poincare conjecture, and was proved by Perelman.