Motivation

In the Aenid, *the Roman poet Virgil tells the story of Queen Dido, the daughter of the Phoenician king of the 9th century B.C.*

After the assassination of her husband by her brother she fled to a haven near Tunis. There she asked the local leader, Yarb, for as much land as could be enclosed by the hide of a bull. Since the deal seemed very modest, he agreed.

Dido cut the hide into narrow strips, tied them together and bounded a large tract of land which became the city of Carthage (located in modern Tunisia).

What shape should Dido make to maximize the area of Carthage?

Math 32A, Calculus of Several Variables
Lecture 1

Richard Wong

Winter 2023, UCLA

Slides can be found on Canvas.

Welcome to 32A!

You belong in this classroom.

- Be comfortable with **asking questions** and **making mistakes**.
  - It's normal to not understand a concept immediately.
- Don't be afraid to **ask for help** if you need it!
  - From me, from your TAs, and from your peers.
- This course emphasizes mathematical **collaboration** and **communication**:
  - **Listen to understand** what the other person is saying.
  - **Challenge the ideas, not the person.**
  - **Be aware** of the space you take up in the discussion.

Syllabus review

- Canvas
- Campuswire **(Code: 4355)**
- **All exams will be held in person, during class.**

Midterm 1	Friday Feb 3	12-12:50pm
Midterm 2	Friday Mar 3	12-12:50pm
Final	Tuesday Mar 21	11:30am-2:30pm

- **Content Office Hours:** Mondays 3-4pm, Fridays 1-2pm, **8251 Boelter Hall**; or by appointment.
- **Social Office Hours:** To Be Announced

Learning Goals

1. You acquire an understanding of the geometry of space, vectors, and the differential calculus of vector functions and multivariable functions.
2. You develop the reasoning and questioning skills needed to explore these (mathematical) topics and apply them to real-life situations.
3. You develop the collaboration and communication skills needed to convey your (mathematical) ideas.

"I don't solve quadratic equations to help me with my daily life, but I do use mathematical thinking to help me understand arguments and to empathize with other people. And so pure maths helps me with the entire human world."

— Eugenia Cheng

Dido's Problem

What shape should Dido make to maximize the area of Carthage?

Dido's problem is also known as the *isoperimetric problem*:

Question

Find, among all curves of given length, the one which encloses maximal area.

By the end of this course, you should be able to **prove** what the answer is.

Gateway to Higher Mathematics

Multivariable calculus is related to several areas in mathematics:

- Linear algebra,
- Ordinary and partial differential equations,
- Real analysis,
- Differential geometry,
- Topology,
- Mathematical modeling,
- Applied mathematics
- and more!

What changes in multivariable calculus? What does it mean for a function to be continuous? To take a derivative? Can we still find maxima and minima? Does the second derivative test still hold?

"In mathematics, the art of asking questions is more valuable than solving problems."

— Georg Cantor

Electromagnetism

Example

Let $\boldsymbol{E}(\boldsymbol{x}, t)$ denote the electric field in $\mathbb{R}^3$, and $\boldsymbol{p}(\boldsymbol{B}, t)$ denote the Magnetic field. Then their behavior is determined by Maxwell's equations

- $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$ (Gauss's law)
- $\nabla \cdot \boldsymbol{B} = 0$ (Gauss's law for magnetism)
- $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ (Faraday's Law)
- $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$ (Ampère-Maxwell Law)

Navier-Stokes Equations

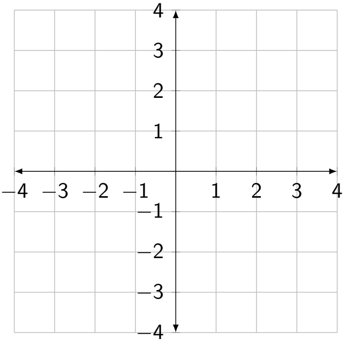
Example (The Navier–Stokes equations)

Let $\boldsymbol{u}(\boldsymbol{x}, t)$ denote the velocity of an incompressible fluid in $\mathbb{R}^3$, and $\boldsymbol{p}(\boldsymbol{x}, t)$ denote the pressure of the fluid. Then the Navier-Stokes equations are:

- $\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla \cdot \boldsymbol{u})\boldsymbol{u} = \nu \Delta \boldsymbol{u} - \nabla \boldsymbol{p} + \boldsymbol{f}(\boldsymbol{x}, t)$
- $\nabla \cdot \boldsymbol{u} = 0$

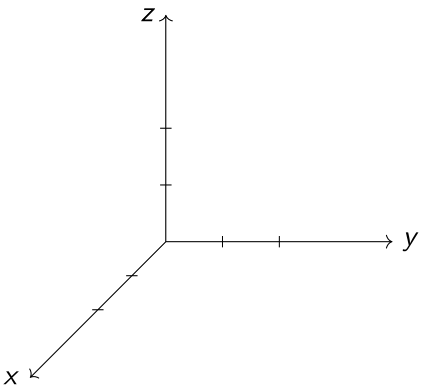
Given initial conditions, solve these equations for $\boldsymbol{u}(\boldsymbol{x}, t)$ and $\boldsymbol{p}(\boldsymbol{x}, t)$. This is one of the **Millenium Prize Problems**.

Recall that the plane $\mathbb{R}^2$ is the set of points $\{(x, y) \mid x, y \in \mathbb{R}\}$.



For example, we can find the point $P = (\pi, e)$.

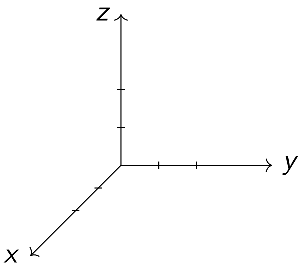
Generalizing, $\mathbb{R}^3$ is the set of points $\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$.



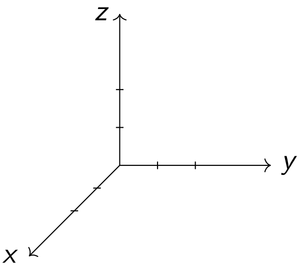
For example, we can find the point $Q = (\pi, e, 3)$.

We can also think of $\mathbb{R}^3$ as a **vector space**. For any point $P \in \mathbb{R}^3$, we can ask (1) how far away is it from the origin, and (2) in what direction?

Thus we obtain a **position vector**, $\boldsymbol{OP}$, which starts at the origin, and terminates at the point P.



In general, a vector $\boldsymbol{v} = \boldsymbol{PQ}$ in $\mathbb{R}^3$ is determined by two points in $\mathbb{R}^3$: an initial point P and a terminal point Q.

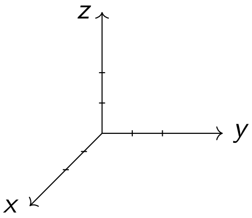


Question

When are two vectors the same?

Definition

Let $\boldsymbol{w} = \boldsymbol{AB}$ and $\boldsymbol{v} = \boldsymbol{PQ}$ be two vectors. If when we move $\boldsymbol{v}$ to start at the point A, without changing its direction or length, we get the vector $\boldsymbol{w}$, then we say that $\boldsymbol{v}$ and $\boldsymbol{w}$ are **equivalent vectors**. $\boldsymbol{w}$ is called a **translation of $\boldsymbol{v}$**.



Proposition

Given a vector $\boldsymbol{v} = \boldsymbol{PQ}$, we can always obtain an equivalent position vector.
That is, a vector that starts at the origin.

Definition

Given a vector $\boldsymbol{v} = \boldsymbol{PQ}$, where $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$, we can write $\boldsymbol{v}$ in terms of its **components**:

$$\boldsymbol{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Question

Let $\boldsymbol{u} = \boldsymbol{PQ}$, where $P = (1, -4, 1)$ and $Q = (-2, -2, 0)$. What is its equivalent position vector?

Question

Let $\boldsymbol{u} = \boldsymbol{PQ}$*, where* $P = (1, -4, 1)$ *and* $Q = (-2, -2, 0)$.

Let $\boldsymbol{v} = \boldsymbol{AB}$*, where* $A = (1, 0, 2)$ *and* $B = (-2, 2, 1)$.

Are $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *equivalent vectors?*

Vote (1) for yes and (2) for no.

Definition

Given a vector $\boldsymbol{v} = \boldsymbol{PQ}$, where $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$, we can write $\boldsymbol{v}$ in terms of its **components**:

$$\boldsymbol{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Theorem

Two vectors $\boldsymbol{v} = \boldsymbol{PQ}$ and $\boldsymbol{w} = \boldsymbol{AB}$ are equivalent if and only if they have the same components.

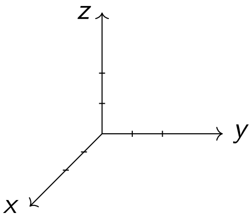
Definition

Given a vector $\boldsymbol{v} = \langle x, y, z \rangle$, its **magnitude** is given by

$$||\boldsymbol{v}|| = \sqrt{x^2 + y^2 + z^2}$$

Question

Given a vector $\boldsymbol{v} = \boldsymbol{PQ}$*, where* $P = (1, 1, 1)$ *and* $Q = (2, 3, 4)$*, what is it's magnitude?*



Question

Given the points $P = (1, 1, 1)$ and $Q = (2, 3, 4)$, what is the distance between P and Q?

Remark

The magnitude of a vector $\boldsymbol{PQ}$ *is the same as the distance between the points P and Q.*

References

Bandle, Catherine
Dido's Problem and Its Impact on Modern Mathematics
Notices of the AMS Volume 64, Number 9, 2017.